UNIFIED ENROLLMENT IN SCHOOL CHOICE: HOW TO IMPROVE STUDENT ASSIGNMENT IN CHICAGO

BATTAL DOĞAN AND M. BUMIN YENMEZ

Abstract. The Chicago Board of Education is implementing a centralized clearinghouse to assign students to schools for 2018-19 admissions. In this clearinghouse, each student can simultaneously be admitted to a selective and a nonselective school. We study this *divided enrollment system* and show that an alternative *unified enrollment system*, which assigns each student to only one school, is better for all students. We also examine systems with two stages of admissions, which has also been considered in Chicago, and establish conditions under which the unified enrollment system is better than the divided enrollment system.

1. Introduction

Market design has been especially successful in assigning students to public schools. In an influential study, Abdulkadiroğlu and Sönmez (2003) introduce market design as a tool in this context, show that some school districts use deficient mechanisms, and propose better alternatives. Since then many school districts including Boston, Denver, Detroit, New Orleans, New York City, Newark, and Washington D.C. have either established centralized clearinghouses or reformed their existing systems.[1]

In April 2017, the Chicago Board of Education voted unanimously to implement a centralized clearinghouse to assign students to schools. Some of these schools are called 'selective' and use standardized testing for admissions, while the rest are 'nonselective,' district-run public schools that may include magnet and neighborhood schools. In the proposed system, each student is admitted to a selective and a nonselective school, and, if the student rejects both offers, then the student may go to a second stage of admissions.[2] We call this the *divided enrollment system* because a student can be assigned to more than one school in the

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Doğan is affiliated with Faculty of Business and Economics (HEC), University of Lausanne, 1015 Lausanne, Switzerland; Yenmez is with Department of Economics, Boston College, 140 Commonwealth Ave, Chestnut Hill, MA, 02467. Emails: battal.dogan@unil.ch, bumin.yenmez@bc.edu.

[1]See Abdulkadiroğlu et al. (2005a) and Abdulkadiroğlu et al. (2005b) for the reforms in Boston and New York City, respectively.

[2]The new admissions policies adopted by the board can be reached at http://www.cpsboe.org/content/actions/2017_04/17-0426-PO2.pdf (retrieved on July 3, 2017). As of now, the second stage is still under consideration. Eventually, it may not be implemented.

clearinghouse. Instead, we propose a *unified enrollment system* in which each student can be assigned to only one school.

In this paper, we study divided and unified enrollment systems for student assignment when there are different types of schools; we show that the unified enrollment system is better for students than the divided enrollment system. Therefore, we show that the planned system in Chicago, in which each student is assigned to a selective and a nonselective school simultaneously, can be improved for all students by moving to the unified enrollment system that assigns each student to only one school. Furthermore, the unified system can be established easily, using the existing mechanisms, with only one change: Instead of eliciting the preferences of students over selective and nonselective schools separately, students' preferences over all schools are elicited.

We assume that the student-proposing deferred-acceptance algorithm (SPDA) of Gale and Shapley (1962) is used to assign students to schools in the unified enrollment system or for each type of school in the divided enrollment system. As far as we know, all of the aforementioned school districts currently use this algorithm. We start by comparing the two enrollment systems when there is only one stage of admissions. We call these benchmark systems the 1-stage divided enrollment system and the 1-stage unified enrollment system. It is well known that SPDA is strategyproof in the sense that no student can get a better school by misreporting his preferences, whatever preferences the other students report (Dubins and Freedman, 1981; Roth, 1982). Therefore, both systems have a truthful equilibrium. We show that either the truthful equilibrium outcome of the unified enrollment system Pareto dominates the truthful equilibrium outcome of the divided enrollment system, or they are the same (Theorem 1). Furthermore, we characterize the sources of inefficiency in the divided enrollment system in Appendix A.

In most of the existing school choice systems, like New York City and Boston, there is a second stage of admissions for unassigned students. In the proposed system in Chicago, a second stage is also mentioned, although it is deemed unnecessary.[3] Therefore, we also study 2-stage enrollment systems. In the 2-stage divided enrollment system, each student is assigned to one school of every school type at the first stage. Then, each student can accept one of the offers or reject all of them and be unassigned. In the second stage, this is repeated for the unassigned students and the remaining schools, which have reduced capacities. This defines an extensive-form game with multiple decision nodes for each student. The 2-stage unified enrollment system is the same, except that each student is assigned to at most one school at each stage. For the 2-stage enrollment systems, it is a dominant strategy to accept the best offer at the second stage, provided that it is better than the outside option for a student. Therefore, we assume that each student follows this strategy. Furthermore, it is still

[3]The report of the Chicago Board of Education states, "There is no guarantee that more than one round of admission will be necessary."

a weakly dominant strategy to submit true preferences at the second stage, so we also assume that each agent follows this truthful strategy. Therefore, in both 2-stage enrollment systems, each agent's strategic decisions are to report preferences and to make acceptance/rejection decisions at the first stage. For some of our results, we further simplify the analysis by focusing on truthful strategies at the first stage.

We analyze the 2-stage divided enrollment system by first assuming that students are strategic only when they make their acceptance/rejection decisions. We show that rejecting all offers weakly dominates accepting the best offer. Therefore, each student rejecting all offers is a Nash equilibrium that produces the same outcome as the truthful equilibrium outcome of the 1-stage divided enrollment system (Theorem 2). To show this result, we provide a new comparative statics of SPDA: When students leave with their allotments, the remaining students are better off in SPDA (Lemma 2). The equilibrium outcome is weakly worse than the truthful equilibrium outcome of the 1-stage unified enrollment system for all students (Corollary 1). However, the 2-stage divided enrollment system can have an equilibrium outcome that Pareto dominates the truthful equilibrium of the 1-stage unified enrollment system (Example 3 in Appendix C). Next we look at the case when schools have efficient capacity-priority structures (Ergin, 2002): a student s is an *interrupter* at school c, if there is a step in the SPDA where student s is tentatively accepted by school c but another student is rejected, while at a later step student s is also rejected (Kesten, 2010). If there is no interrupter in the 1-stage unified enrollment system, then the capacity-priority structure is *efficient for unified enrollment*. If there is no interrupter at the 1-stage divided enrollment system, then the capacity-priority structure is *efficient for divided enrollment.* We show that every student is weakly better off at the truthful equilibrium of the 1-stage unified enrollment system compared to any Nash equilibrium outcome of the 2-stage divided enrollment system (Theorem 3). Second, we consider the case when students are also strategic when reporting first-stage preferences. In this case, we show that when the capacity-priority structure is efficient for divided enrollment, then every student is weakly better off in the truthful outcome of the 1-stage unified enrollment system compared to any subgame perfect Nash equilibrium (SPNE) of the 2-stage divided enrollment system (Theorem 4).

Finally, we compare equilibria of the 2-stage unified enrollment system and the 2-stage divided enrollment system. In both systems, rejecting all offers is a weakly-dominant strategy. Therefore, it is an equilibrium when students reject all offers (Theorem 5). However, this is the worst equilibrium for the students (Theorem 6). Students weakly prefer the equilibrium outcome under the 2-stage unified enrollment system when they reject all offers at the first stage to the equilibrium outcome under the 2-stage divided enrollment system when all offers are rejected (Theorem 5). Therefore, we can compare the worst equilibria for students under two 2-stage enrollment systems. However, we cannot compare the best equilibria for students, as such an equilibrium may not exist for the 2-stage divided enrollment system (Example 6 in Appendix C). Furthermore, when the capacity-priority structure is efficient for unified enrollment accepting the offer and rejecting it after the first stage are payoff equivalent for every student in the 2-stage unified enrollment system. Therefore, every pure strategy profile is a Nash equilibrium that produces the same outcome as the truthful equilibrium outcome of the 1-stage unified enrollment system. This equilibrium outcome is weakly better for all students than any equilibrium of the 2-stage divided enrollment system (Theorem 7). In addition, we consider the case when students are also strategic when reporting first-stage preferences. If the capacity-priority structure is efficient for unified enrollment, then there exists a SPNE of the 2-stage unified enrollment system that produces the same outcome as the truthful equilibrium outcome of the 1-stage unified enrollment system (Theorem 8). This equilibrium outcome is weakly better for all students than any SPNE of the 2-stage divided enrollment system, which follows from Theorem 4.

Related Literature. An important feature of our model is the coexistence of two centralized enrollment systems involving the same set of students. Manjunath and Turhan (2016) also consider a model with the same feature and show that the 1-stage divided enrollment system cannot always avoid waste (in their Theorem 1) and cannot always coincide with the 1-stage unified enrollment outcome (in their Corollary 1). Our Theorem 1 shows that the 1-stage unified enrollment Pareto improves over the 1-stage divided enrollment system.

Another important feature of our model is that the enrollment system may run in more than one round. Manjunath and Turhan (2016) also consider a multi-stage divided enrollment system. However, they assume that at each round the matching is tentative, so all students participate. Furthermore, at any round, each student truncates his preferences at the school he was assigned in the previous round without any strategic concerns. In contrast, a first-stage acceptance by a student is permanent in our 2-stage divided enrollment system like the planned system in Chicago. Furthermore, students are strategic in our setting. Another paper that considers student assignment in multiple rounds is Dur and Kesten (2014). Their model, unlike ours, does not feature divided enrollment, and their main focus is on understanding whether any sequential enrollment system satisfying certain design goals (such as non-wastefulness, strategy-proofness, fairness, and respecting improvements) exists, while we take the mechanism to be used at each round of the sequential process as given (SPDA) and compare the divided enrollment system with the unified enrollment system.

In a recent work, Ekmekci and Yenmez (2014) introduce a framework to study the incentives of schools to join a centralized clearinghouse and provide an explanation of why some charter schools have evaded (or may evade) the clearinghouse. When a school evades the centralized clearinghouse, it may run its admissions simultaneously, before, or after the centralized clearinghouse. The extensive-form games in Ekmekci and Yenmez (2014) are quite different than the games we use to analyze the divided enrollment systems. More importantly, our focus is student welfare, whereas they study school incentives.

2. Model

Let $\mathcal{S} = \{s_1, \ldots, s_{|\mathcal{S}|}\}$ be a set of **students** and $\mathcal{C} = \{c_1, \ldots, c_{|\mathcal{C}|}\}$ be a set of **schools**. Each school has a **type**. In our motivating Chicago Public School application, there are two school types: *selective* and *nonselective*. In general, there are $K \in \mathbb{N}$ school types. The set of type-k schools is denoted by $\mathcal{C}_k$ for each type k. Therefore, $\{\mathcal{C}_1, \ldots, \mathcal{C}_K\}$ is a partition of the set of schools $\mathcal{C}$.

Each student $s \in \mathcal{S}$ has a **preference relation** R_s over $\mathcal{C} \cup \{s\}$,[4] where s represents an outside option for the student, which can be the neighborhood school, private school, or homeschooling. Given $c, c' \in \mathcal{C} \cup \{s\}$, we write $c\ P_s\ c'$ if $c \neq c'$ and $c\ R\ c'$, i.e., student s strictly prefers school c to school c'. A school c is **acceptable** to student s if the school is strictly better than the outside option, or equivalently, $c\ P_s\ s$.

Each school $c \in \mathcal{C}$ has a **capacity** $q_c \in \mathbb{N}$, which represents the maximum number of students the school can admit and a **priority ranking** $\succeq_c$ over the set of students $\mathcal{S} \cup \{c\}$.[5] Here, c denotes the option of having an empty seat. The strict part of the priority ranking $\succeq_c$ is denoted as $\succ_c$, so, for any $s, s' \in \mathcal{S} \cup \{c\}$, if $s \succeq_c s'$ and $s \neq s'$ then $s \succ_c s'$. A student s is **acceptable** to school c if the student is strictly better than the outside option, or equivalently, $s \succ_c c$. For some of the public schools, all students may be acceptable, whereas for others, like the selective schools in Chicago or some magnet schools, which may require an entrance examination, interview, or audition, some of the students may be unacceptable.

The admission policy of each school $c \in \mathcal{C}$ is represented by a **choice rule** $Ch_c : \mathcal{S} \times \{1, \ldots, q_c\} \to \mathcal{S}$, which maps each nonempty set $S \subseteq \mathcal{S}$ of students to a subset $Ch(S; q) \subseteq S$ of chosen students such that $|Ch(S; q)| \leq q$. Here, q represents the remaining capacity, so the school cannot admit more than q students. In a 2-stage enrollment system, the admissions policy may depend on the number of seats available since a school needs to specify how to allocate the remaining seats at the second stage after some of the seats are allocated in the first stage. We assume that for each school $c \in \mathcal{C}$, Ch_c is **responsive** to the priority ranking $\succeq_c$, i.e., for each $S \subseteq \mathcal{S}$, $Ch_c(S; q)$ is obtained by choosing the highest-priority acceptable students in S until q students are chosen or no acceptable student is left.

[4]More formally, a preference relation over $\mathcal{C} \cup \{s\}$ is a complete, transitive, and anti-symmetric binary relation over $\mathcal{C} \cup \{s\}$. Binary relation R_s over $\mathcal{C} \cup \{s\}$ is *complete* if, for every $c_1, c_2 \in \mathcal{C} \cup \{s\}$, $c_1 R_s c_2$ or $c_2 R_s c_1$. It is *transitive* if, for every $c_1, c_2, c_3 \in \mathcal{C} \cup \{s\}$, $c_1 R_s c_2$ and $c_2 R_s c_3$ imply $c_1 R_s c_3$. It is *anti-symmetric* if, for every $c_1, c_2 \in \mathcal{C} \cup \{s\}$, $c_1 R_s c_2$ and $c_2 R_s c_1$ imply $c_1 = c_2$.

[5]The priority ranking $\succeq_c$ is a complete, transitive, and anti-symmetric binary relation over $\mathcal{S} \cup \{c\}$.

A **market** is a tuple $(\mathcal{S}, (\mathcal{C}_k)_{k\in\{1,\ldots,K\}}, (R_s)_{s\in\mathcal{S}}, (q_c, \succeq_c)_{c\in\mathcal{C}})$. Sometimes, we fix all the primitives of a market other than the student preference relation profile; in that case, we refer to the student preference relation profile $(R_s)_{s\in\mathcal{S}}$ as the market.

A **matching** μ is an assignment of students to schools respecting the capacity constraints:

- for each student s, $\mu(s) \in \mathcal{C}$ or $\mu(s) = s$,
- for each school c, $\mu(c) \subseteq \mathcal{S}$ and $|\mu(c)| \leq q_c$, and
- for each student s and school c, $\mu(s) = c$ if, and only if, $s \in \mu(c)$.

In words, every student is either matched with a school or unmatched, and every school gets a set of students with cardinality less than or equal to its capacity. In addition, there is a feasibility constraint such that if a student gets matched with a school, then the student is in the set of students matched with the school.

A matching μ is **stable** if it satisfies the following properties:

- (individual rationality for students) for each student s, $\mu(s)\ R_s\ s$,
- (individual rationality for schools) for each school c, $Ch_c(\mu(c); q_c) = \mu(c)$, and
- (no blocking) there exist no student-school pair (s, c) such that $c\ P_s\ \mu(s)$ and $s \in Ch_c(\mu(c) \cup \{s\}; q_c)$.

A matching μ **Pareto dominates** another matching ν, if for every student s, $\mu(s)$ is weakly better than $\nu(s)$ and for one student it is strictly better. A matching that is not Pareto dominated is called **Pareto efficient**; otherwise, it is **Pareto inefficient**.

Individual rationality for a student means that she prefers the outcome over being unmatched. On the other hand, for a school, it means that the school would like to keep all the students assigned to it. More explicitly, since choice rules are responsive, only acceptable students are matched with the school and the number of students is no more than the capacity of the school. No blocking rules out the existence of a student-school pair such that the student likes the school more than her match and the school would like to admit the student.

In student-assignment settings, stability of a matching is viewed as a fairness notion (Abdulkadiroğlu and Sönmez, 2003). Furthermore, the deferred-acceptance algorithm of Gale and Shapley (1962) is implemented in school districts that have reformed their student assignment mechanisms (Abdulkadiroğlu et al., 2005a,b). The following describes how the algorithm works for a set of students S and a set of schools C. We assume that each school c's available capacity is its quota q_c, but the algorithm can be applied for any remaining capacity.

Student-Proposing Deferred Acceptance Algorithm (SPDA)

Step 1. Each student $s \in S$ applies to her top-ranked acceptable school in C. If there is no such school, then she is unmatched. Each school c considers its applicants, say $A_1(c)$.

Among these, it tentatively accepts $Ch_c(A_1(c); q_c)$. It rejects all other applicants. If there is no rejection by any school at this step, then stop.

Step $t \geq 2$. Each student s who is rejected at Step $t-1$ applies to her top-ranked acceptable school among the ones that have not rejected her. If there is no such school, she is unmatched. Each school c considers the students that it tentatively accepted at Step $t-1$ and the new applicants at Step t, say $A_t(c)$. Among these, it tentatively accepts $Ch_c(A_t(c); q_c)$. It rejects all other applicants. If there is no rejection by any school at this step, then stop.

The algorithm stops in finite time since there can only be a finite number of rejections.

3. Enrollment Systems and Games

The unified enrollment system is used in various school districts in the United States, as well as in college admissions in some countries around the world where students are matched with schools (or colleges) in a centralized clearinghouse. However, even though the proposed student assignment system in Chicago is centralized, it is *divided* in the sense that each student can be admitted to a selective school and a nonselective school simultaneously. Likewise, there is a debate whether the student assignment should be done in two stages or just one. On the other hand, the student assignment systems in Boston and New York City have supplementary rounds for unassigned students. Thus, we consider four centralized enrollment systems depending on whether there exists a universal enrollment system or not and whether the system has two stages or only one stage.

1- Stage Unified Enrollment System: Each student reports a preference relation over *all* schools. Then a matching is determined by running SPDA. Therefore, each student is matched to at most one school. Finally, each student either accepts his match or rejects it (in which case he is unmatched or matched to his outside option).

2-Stage Unified Enrollment System: In the first stage, a matching is determined through the 1-stage unified enrollment system described above. In the second stage, if there are any remaining students (students who were unmatched or did not accept their matches at the end of the first stage) and there are available seats, another matching is determined through the 1-stage unified enrollment system among the remaining students and schools with available seats (note that those students who accepted their matched schools in the previous stage leave with their allotments and do not participate in the second stage).

1-Stage Divided Enrollment System: Each student reports, for each school type, his preference relation over the schools of that type, so he reports K separate preference relations. Then, a matching for each school type is determined by running SPDA. As a result, each student may be matched to one school of every type (in the Chicago Public School application each student may be matched with a selective school and a nonselective school). Finally, each student accepts at most one school from her matches (possibly rejecting all of them).

Formally, first, each student $s \in \mathcal{S}$ reports a profile of K preference relations, $(R_s^k)_{k\in\{1,\ldots,K\}}$, where for each $k \in \{1, \ldots, K\}$, R_s^k is a preference relation of student s over set of schools $\mathcal{C}_k \cup \{s\}$. Then, for each $k \in \{1, \ldots, K\}$, students are matched to schools in $\mathcal{C}_k$ using SPDA. Let μ_k denote the resulting matching. Finally, each student s chooses at most one school (possibly none) from $\cup_{k\in\{1,\ldots,k\}}\{\mu_k(s)\}$.

2-Stage Divided Enrollment System: In the first stage, a matching is determined through the 1-stage divided enrollment system described above. In the second stage, if there are any remaining students (students who were unmatched or did not accept any of their matches at the end of the first stage) and there are available seats, another matching is determined through the 1-stage divided enrollment system among the unmatched students and schools with available seats (note that those students who accepted one of their offers in the previous stage leave with their allotments and do not participate in the second stage).

We assume that students may be strategic when reporting their preferences or making their accept/reject decisions among schools that have admitted them, but schools are not strategic, and their priority rankings and quotas are common knowledge. Each enrollment system, at each preference profile, induces an *extensive-form game*. We study the Nash equilibrium and SPNE of these games and compare the equilibria of the different enrollment systems.

Note that, in the extensive-form game at a given preference profile, a student's strategy should recommend, at each stage, what preference relation to report for each school type *as a function of the history* and which matched schools to accept *as a function of the history*. Let us call a student's strategy **truthful** if it recommends reporting, at each stage, for each school type, and for each history, the true preference relation over schools. Note that there can be many different truthful strategies, which differ in the acceptance-rejection recommendations but always report the preferences truthfully.

We say that a student's strategy **weakly dominates** another strategy if, for each strategy profile of the other students, the former strategy yields an outcome that is weakly preferred by the student to the outcome of the latter strategy.[6] We say that two student strategies are **payoff equivalent** if, for each strategy profile of the other students, the student is indifferent between the outcomes of the two strategies.

[6]This is the definition of weak domination used in mechanism design (Jackson, 2003) and matching theory (Roth and Sotomayor, 1990, Definition 4.2). However, in the game-theory literature, weak domination also requires that there exists a strategy profile of the other students for which the outcome of the former strategy is strictly preferred to the outcome of the latter strategy.

It is a weakly dominant strategy for each student to truthfully report her preferences in SPDA (Dubins and Freedman, 1981; Roth, 1982). In other words, this algorithm has a truthful *Nash equilibrium with weakly-dominant strategies.* As a result, most of the literature focuses on this particular equilibrium, even though it may have other Nash equilibria.[7] Following the literature, we consider this truthful equilibrium in the one-stage admissions systems. Furthermore, in the two-stage admissions systems, we study SPNE in which students are truthful at the second stage.

4. 1-Stage Unified Enrollment vs. 1-Stage Divided Enrollment

Under both one-stage enrollment systems, in any SPNE, each student accepts the best offer from acceptable schools. Then, in the unified enrollment system, truthfully reporting preferences remains a weakly dominant strategy. Likewise, in the 1-Stage Divided Enrollment System, each student has a weakly dominant strategy where he truthfully reports his preferences over each school type because SPDA is used for each school type, and there is no interaction between the outcomes among different school types. We focus on this equilibrium of both enrollment systems and show that each student prefers the 1-stage unified enrollment system over the 1-stage divided enrollment system.

Theorem 1. *Every student is weakly better off in the truthful equilibrium of the 1-stage unified enrollment system compared to the truthful equilibrium of the 1-stage divided enrollment system.*

Intuitively, there is more competition for every school type in the 1-stage divided enrollment system compared to the 1-stage unified enrollment system. This intuition is formalized in Lemma 1 in Appendix B, which contains all omitted proofs. As a result, each student is weakly worse off in the 1-stage divided enrollment system than the 1-stage unified enrollment system.

Theorem 1 shows that the 1-stage divided enrollment system is not efficient. There are multiple reasons for the inefficiency. One obvious reason is that some of the seats may be wasted, as a student can get into two or more schools but can accept only one. However, the inefficiency of the divided enrollment system is not limited to this waste. In Appendix A, we characterize the sources of inefficiency for the divided system.

[7]This equilibrium has been commonly used in the literature in the analysis of SPDA. For instance, Ergin and Sönmez (2006) analyze Nash equilibrium outcomes of the game induced by another well-known system, the Boston mechanism, and compare those outcomes with the truthful equilibrium outcome of SPDA. Yet, SPDA may have other Nash equilibria, even with an unstable outcome (see Sotomayor, 2008). Furthermore, Bando (2014) shows that the outcome of the efficiency-adjusted deferred-acceptance mechanism (Kesten, 2010) is a strict Nash equilibrium of SPDA, which is weakly better than truthful equilibrium outcome of SPDA.

5. 1-Stage Unified Enrollment vs. 2-Stage Divided Enrollment

The proposed system in Chicago has two stages. At the first stage, students submit their preferences over selective schools and nonselective schools. Then SPDA is used separately for each school type. Each student can either accept one offer, if she has been admitted to at least one school, or reject all offers. At the second stage, students submit new preferences and the same process is repeated. If a student has accepted an offer, that is her assignment; otherwise, the student is assigned to her neighborhood school, which is represented by the outside option in our model.

First, we consider the case when students use truthful strategies.

5.1. Students are strategic only in acceptance-rejection decisions. Suppose that each student reports his preferences truthfully in the first stage, and also in the second stage, if he rejects all his offers at the end of the first stage. More formally, we assume that students are truthful. In addition, accepting the best offer after the second stage is optimal for students, so the only strategic decisions for students are in the acceptance-rejection decisions at the end of the first stage.

Consider the game where students observe their assigned schools at the first stage and decide whether to accept an offer or not, and students who reject their offers move to the second stage, where they report truthfully and accept the best offer in the end. At each market, this game can be represented by a one-stage simultaneous-move game where the strategy set of each student consists of accepting an offer or rejecting all offers. Moreover, for each student, accepting an offer which is not his best offer is strictly dominated (by, for example, accepting the best offer). Thus, we can simply focus on the game where each student has two strategies: accept the best offer or reject all offers. The next result shows that rejecting all offers weakly dominates accepting the best offer, so we do not need to define information sets of students.

Theorem 2. *In the 2-stage divided enrollment system, for each student, rejecting all offers weakly dominates accepting the best offer. Consequently, each student rejecting all offers is a Nash equilibrium, which produces the same outcome as the benchmark 1-stage divided enrollment system when students are truthful.*

To show this result, we prove a property of SPDA in Appendix B (Lemma 2). Run SPDA once. Let a set of students leave the market with their assigned seats in SPDA, reducing the capacities of the matched schools. Run SPDA again in the reduced market. In this setting, it is known that the matching for the remaining students may change, unless we make assumptions on school priorities and capacities (Ergin, 2002).[8] Instead, we show that

[8] In other words, SPDA does not satisfy the consistency property studied in Thomson (1990). In fact, no stable mechanism satisfies consistency (Afacan and Dur, 2017).

each student remaining in the market gets a weakly better school. Lemma 2 shows one of two effects that a leaving student may have on the remaining students. The second effect is that when a student is matched with a school of a particular type and leaves the market, there is less competition for other types of schools. This follows from comparative statistics of SPDA (Kelso and Crawford, 1982; Chambers and Yenmez, 2013). These two effects imply that rejecting all offers after the first stage is a weakly dominant strategy since at the second stage every student is matched with the same school or a better one for every school type, regardless of what other students decide.

Remark 1. Example 1 in Appendix C shows that, for a student, rejecting all offers may be payoff equivalent to accepting the best offer. The example also exhibits a Nash equilibrium outcome of the 2-stage divided enrollment system that is the same as the 1-stage unified enrollment outcome.

An implication of Theorem 2 is the following:

Corollary 1. *When students are truthful, every student is weakly better off in the 1-stage unified enrollment system compared to the Nash equilibrium of the 2-stage divided system in which students reject all offers after the first stage.*

Remark 2. In addition, there are markets in which the truthful 1-stage unified enrollment outcome Pareto dominates every Nash equilibrium outcome of the 2-stage divided enrollment system. See Example 2 in Appendix C. However, a Nash equilibrium outcome of the 2-stage divided enrollment system may Pareto dominate the 1-stage unified enrollment outcome too. Example 3 in Appendix C demonstrates this.

We next study when the 1-stage unified enrollment outcome is always weakly better than all the equilibria of the 1-stage divided system. To do this, we need the following definitions.

Given a market $(R_s)_{s\in\mathcal{S}}$, say that student s is an **interrupter** at school c if there is a step at SPDA where student s is tentatively accepted by school c but another student is rejected, while at a later step student s is also rejected (Kesten, 2010). Note that according to this definition, student s could be accepted by the school for the first time at an earlier step without rejecting any student, but the school may reject another student at a later step while still accepting student s. In this case, student s would still be an interrupter if she is rejected at a later step.

We say that the capacity-priority structure $(q_c, \succeq_c)_{c\in\mathcal{C}}$ is **efficient for unified enrollment** if in any market, there is no interrupter at any school in the 1-stage unified enrollment system. We say that the capacity-priority structure $(q_c, \succeq_c)_{c\in\mathcal{C}}$ is **efficient for divided enrollment** if in any market, there is no interrupter at any school in the 1-stage divided enrollment system. The efficient capacity-priority assumption is introduced in Ergin (2002) as *acyclicity*.[9] He shows that SPDA is Pareto efficient if, and only if, the capacity structure is efficient (for unified enrollment). It is easy to see that if a capacity-priority structure is efficient for unified enrollment, then it is also efficient for divided enrollment, but not vice versa.

Theorem 3. *Suppose that the capacity-priority structure is efficient for divided enrollment. Then every student is weakly better off in the truthful outcome of the 1-stage unified enrollment system compared to any Nash equilibrium of the 2-stage divided enrollment system.*

To show this result, we first show that a Nash equilibrium outcome of the 2-stage divided enrollment system cannot have a blocking pair. The intuition is that, if the priority of student s is violated by another student s' at a school c, then students s and s' must be receiving their final assignments in markets at different stages of the two-stage divided enrollment system. Moreover, student s, by a unilateral deviation, can instead participate in the market where student s' receives school c, and can guarantee himself at least school c, since a lower-priority student is receiving c before his arrival into the market. Here, the efficient capacity-priority structure assumption is crucial because otherwise student s could have been an interrupter in the new market and could have ended up with a school worse than school c, although without his presence a lower priority student is able to receive school c. Once we show that there is no blocking pair, the rest follows from the fact that the 1-stage unified enrollment outcome is the student-optimal stable outcome.

5.2. Students are also strategic when reporting first-stage preferences. Suppose now that students are also strategic when reporting their first-stage preferences, in addition to acceptance-rejection decisions at the end of the first stage. We assume that a student truthfully reports in the second stage if he rejects all of his first-stage offers.

Consider the complete information game where students report preferences (for each school type), then observe the reported preferences at the first stage (and also the profile of offers), and decide whether to accept an offer or not, and the students who reject their offers move to the second stage, where they report truthfully and accept the best offer in the end. At each problem R, this game can represented by an extensive-form complete-information game where a student strategy is a complete contingent plan that specifies which preferences to report and which offer to accept (if any) following each possible reported preference profile. It is easy to see that for each student, accepting an offer which is not his best offer is strictly dominated (by, for example, accepting the best offer). Thus, we can simply focus on the game where a student strategy specifies which preferences to report and one of the two actions to either accept the best offer or reject all offers for each possible reported preference

[9]Ergin's definition does not use interrupters or SPDA; instead, it is a direct restriction on the capacity-priority structure. See Definition 1 in Ergin (2002).

profile. We compare the SPNE of this game with the truthful equilibrium of the 1-stage unified enrollment system.

Remark 3. We start by making two observations. First, there exists a market in which the 2-stage divided enrollment system does not have a pure-strategy SPNE (Example 4 in Appendix C). Second, a SPNE outcome of the 2-stage divided enrollment system may Pareto dominate the truthful equilibrium outcome of the 1-stage unified enrollment system (Example 5 in Appendix C).

Theorem 4. *In any market with an efficient capacity-priority structure for divided enrollment, no student is better off at any SPNE of the 2-stage divided enrollment system compared to the truthful equilibrium of the 1-stage unified enrollment system.*

In particular, a SPNE outcome of the 2-stage divided enrollment system is either the same as the truthful equilibrium outcome of the 1-stage unified enrollment system, or it is Pareto dominated by the latter. The intuition behind this result, and also the structure of its proof, are very similar to those of Theorem 3, except that we need to consider possible one-shot deviations at more decision nodes (in particular, at the first stage preference revelation decision node).

6. 2-Stage Unified Enrollment vs. 2-Stage Divided Enrollment

In this section, we compare the unified and divided enrollment systems when they both have two stages.

6.1. Students are strategic only in their acceptance-rejection decisions. First, we consider the case when students are only strategic in the acceptance-rejection decisions after the first step. Students use truthful strategies, and they accept the best offer after the second stage.

Theorem 5. *Rejecting all offers is a weakly-dominant strategy in both the 2-stage unified and divided enrollment systems. Therefore, both systems have an equilibrium in which students reject all offers. For all students, the equilibrium outcome in the 2-stage unified enrollment system when students reject their offers is weakly better than the equilibrium outcome in the 2-stage divided enrollment system when students rejects all offers.*

Theorem 5 follows from the previous results that we have established. When a student rejects all offers after the first stage, she is guaranteed to get better matches, either because there is less competition in the second stage or because students may have taken their offers (Lemma 2). As a result, the strategy profile in which students reject all offers is a Nash equilibrium in weakly-dominant strategies. Furthermore, for each enrollment system, the equilibrium outcome in the 2-stage process is equivalent to the truthful equilibrium outcome in the 1-stage process. Then each student prefers the unified system as there is more competition for each school in the divided system (Theorem 1, which follows from Lemma 1).

Besides all students rejecting offers being a weakly-dominant strategy equilibrium, another justification for focusing on this particular equilibrium to compare the two 2-stage enrollment systems is the following. All students rejecting offers is the worst equilibrium for students under both systems, as we state below in Theorem 6. Therefore, the comparison in Theorem 5 is a worst-case scenario comparison, while a similar comparison cannot be done based on the best-case scenarios, since the 2-stage divided enrollment system may not have a best equilibrium for students, as shown in Example 6 in Appendix C.

Theorem 6. *The weakly-dominant strategy equilibrium of rejecting all offers is the worst equilibrium for all students simultaneously, both in the 2-stage unified and divided enrollment systems.*

This result follows from the comparative statics result that students leaving with their schools in SPDA improve the outcome for the remaining students (Lemma 2) and the resource monotonicity property that each remaining student benefits when students leave the market unmatched (Kelso and Crawford, 1982; Chambers and Yenmez, 2013). For a student, an equilibrium outcome is better than the alternative strategy profile in which the only difference is that the student rejects all offers (by definition of equilibrium). However, this alternative strategy profile produces an outcome that is weakly better than the outcome under the strategy profile in which all the students reject offers after the first stage (by the two comparative statics). The result follows.

Theorem 7. *Suppose that the capacity-priority structure is efficient for unified enrollment. Then, for every student, accepting the offer and rejecting it after the first stage are payoff equivalent in the 2-stage unified enrollment system. Therefore, every strategy profile is a Nash equilibrium. Furthermore, all equilibria of the 2-stage unified enrollment system produce the same matching as the truthful equilibrium of the 1-stage unified enrollment system. For all students, the equilibrium outcome of the 2-stage unified enrollment system is weakly better than any equilibrium of the 2-stage divided enrollment system.*

Remark 4. Let a **matching mechanism** take an integration market and produce a matching. A matching mechanism φ is **consistent** if a set of students take their allotments under φ and leave, then the mechanism φ keeps the matchings the same for the reduced market (Thomson, 1990). A restriction on the capacity-priority structure $(q_c, \succeq_c)_{c\in\mathcal{C}}$ **implies consistency** for a mechanism if the mechanism is consistent under this restriction. Theorem 7 holds for any 2-stage unified enrollment system in which a matching mechanism φ is used under a capacity-priority structure that implies consistency for the mechanism φ.

When the capacity-priority structure is efficient for unified enrollment, SPDA satisfies consistency (Ergin, 2002). As a result, each student gets the same school whether he accepts the offer after the first stage or rejects it, regardless of what other students do. In other words, accepting the offer or rejecting it are payoff equivalent. This implies that all strategies form an equilibrium and that they produce the truthful SPDA outcome. But this outcome is weakly better for all students than any equilibrium outcome of the 2-stage divided enrollment system (Theorem 3).

6.2. Students are also strategic when reporting first-stage preferences. Finally, we compare the 2-stage enrollment systems when students are also strategic in reporting their first-stage preferences. We establish that the unified system is better.

Theorem 8. *Suppose that the capacity-priority structure is efficient for unified enrollment. If there exists a SPNE of the 2-stage unified enrollment system, then there exists a SPNE of the 2-stage unified enrollment system that produces the same outcome as the truthful equilibrium outcome of the 1-stage unified enrollment system. This equilibrium outcome is weakly better for all students than any SPNE outcome of the 2-stage divided enrollment system.*

To establish that there exists a SPNE that produces the truthful equilibrium of SPDA, we assume that the 2-stage unified enrollment system has *one* SPNE. Without this assumption, we can show that there exists a NE, not necessarily a SPNE, which implements the same outcome. This equilibrium outcome is weakly better for students than any SPNE of the 2-stage divided enrollment system (Theorem 4).

7. Conclusion

Motivated by the recent decision of the Chicago Board of Education to implement a divided enrollment system, we have introduced a new model of school choice. We have shown that the alternative unified enrollment system is weakly better for all students than the divided enrollment system when they both have only one stage of admissions. We have extended our analysis to the case when there are two stages of admissions under reasonable assumptions on school capacity-priority structures.

References

Abdulkadiroğlu, Atila and Tayfun Sönmez, "School choice: A mechanism design approach," *American Economic Review*, June 2003, *93* (3), 729–747.

———, **Parag Pathak, Alvin Roth, and Tayfun Sönmez**, "The Boston Public School Match," *American Economic Review*, May 2005, *95* (2), 368–371.

Abdulkadiroğlu, Atilla, Parag A. Pathak, and Alvin E. Roth, "The New York City High School Match," *American Economic Review*, 2005, *95* (2), pp. 364–367.

Afacan, Mustafa Oğuz and Umut Mert Dur, "Incompatibility between stability and consistency," *Economics Letters*, 2017, *150*, 135 – 137.

Bando, Keisuke, "On the existence of a strictly strong Nash equilibrium under the student-optimal deferred acceptance algorithm," *Games and Economic Behavior*, 2014, *87*, 269–287.

Chambers, Christopher P. and M. Bumin Yenmez, "Choice and matching," 2013. Forthcoming in American Economic Journal: Microeconomics.

Dubins, Lester E and David A Freedman, "Machiavelli and the Gale-Shapley algorithm," *The American Mathematical Monthly*, 1981, *88* (7), 485–494.

Dur, Umut and Onur Kesten, "Sequential versus Simultaneous Assignment Systems and Two Applications," 2014. Working Paper.

Ekmekci, Mehmet and M. Bumin Yenmez, "Integrating Schools for Centralized Admissions," 2014. Working Paper.

Ergin, Haluk and Tayfun Sönmez, "Games of school choice under the Boston mechanism," *Journal of Public Economics*, 2006, *90*, 215–237.

Ergin, Haluk I., "Efficient resource allocation on the basis of priorities," *Econometrica*, 2002, *70* (6), 2489–2497.

Gale, David and Lloyd S. Shapley, "College Admissions and the Stability of Marriage," *The American Mathematical Monthly*, January 1962, *69* (1), 9–15.

Jackson, Matthew O., "Mechanism Theory," in Ulrich Derigs, ed., *Optimization and Operations Research, Encyclopedia of Life Support Systems*, Vol. 3, Oxford, U.K.: EOLSS, 2003.

Kelso, Alexander S. and Vincent P. Crawford, "Job Matching, Coalition Formation, and Gross Substitutes," *Econometrica*, 1982, *50*, 1483–1504.

Kesten, Onur, "School choice with consent," *The Quarterly Journal of Economics*, 2010, *125* (3), 1297–1348.

Manjunath, Vikram and Bertan Turhan, "Two school systems, one district: What to do when a unified admissions process is impossible," *Games and Economic Behavior*, 2016, *95*, 25 – 40.

Roth, Alvin E., "The Economics of Matching: Stability and Incentives," *Mathematics of Operations Research*, 1982, *7*, 617–628.

——— **and Marilda Sotomayor**, *Two-sided Matching: A Study in Game-Theoretic Modelling and Analysis*, Vol. 18 of *Econometric Society Monographs*, Cambridge University Press, Cambridge England, 1990.

Sotomayor, Marilda, "The stability of the equilibrium outcomes in the admission games induced by stable matching rules," *International Journal of Game Theory*, 2008, *36*, 621–640.

Thomson, William, "The Consistency Principle," in Tatsuro Ichiishi, Abraham Neyman, and Yair Tauman, eds., *Game Theory and Applications*, New York: Academic Press, 1990, pp. 187–215.

Appendix A. Sources of Inefficiency

Theorem 1 shows that the 1-stage divided enrollment system results in an outcome that is worse than the unified enrollment outcome for students. In this appendix, we identify three sources of inefficiency resulting from a divided enrollment system.

Given a market, let $S \subseteq \mathcal{S}$ be the set of students who are matched to different schools at the truthful equilibrium of the 1-stage divided enrollment system and the truthful equilibrium of the 1-stage unified enrollment system. Note that, students in S are worse off in the divided enrollment system by Theorem 1, i.e., they *suffer* from divided enrollment. Let μ^u denote the truthful equilibrium outcome of the 1-stage unified enrollment system and let μ^d denote the truthful equilibrium outcome of the 1-stage divided enrollment system.

The first source of inefficiency is due to *waste*. A student who has a relatively higher priority at both selective and nonselective schools may receive offers from two schools, and a seat at the school that he ends up rejecting may be wasted if there is another student who prefers that school to schools that make him offers. In fact, the waste can hurt a chain of students if the student who loses his seat due to waste ends up taking the seat of another student at another school, and so on. We formalize this below.

Say that a student $s \in S$ **suffers from waste** if there exists a list of students $(s_1, \ldots, s_n)$ such that $s_1 = s$ and

(1) in the 1-stage divided enrollment system, each student in the list receives the assignment of the preceding student in unified enrollment school, i.e., for each $i \in \{2, \ldots, n\}$, $\mu^d(s_i) = \mu^u(s_{i-1})$,
(2) for each student in the list except student s_n, the school that he is assigned to in the unified enrollment system does not have an empty seat in the 1-stage divided enrollment system, i.e., for each $i \in \{1, \ldots, n-1\}$, if $\mu^u(s_i) = c$, then $|\mu^d(c)| = q_c$, and
(3) the school that student s_n is matched to in the unified enrollment system has an unassigned seat at the 1-stage divided enrollment outcome, i.e., if $\mu^u(s_n) = c$, then $|\mu^d(c)| < q_c$.

The second source of inefficiency is due to *miscoordination*. Suppose that there are two students s and s' such that student s has a higher priority at a selective school and student s' has a higher priority at a nonselective school. Suppose also that student s prefers the nonselective school and student s' prefers the selective school. In the 1-stage divided enrollment system, they may end up applying to both schools and block each other from receiving better schools, while the unified enrollment system would have *coordinated* their preferences and they would not have applied to each other's more preferred schools. In fact, this type of inefficiency can also happen through a longer cycle of students as follows.

Say that a student $s \in S$ **suffers from miscoordination** if there exists a list of students $(s_1, \ldots, s_n)$ such that $s_1 = s$ and

(1) in the 1-stage divided enrollment system, each student in the list receives the school of the preceding student in the unified enrollment system, i.e., for each $i \in \{2, \ldots, n\}$, $\mu^d(s_i) = \mu^u(s_{i-1})$, and student s_1 receives the school that student s_n receives in the unified enrollment system, that is, $\mu^d(s_1) = \mu^u(s_n)$,
(2) for each student the school that she receives in the unified enrollment system does not have empty seats in the 1-stage divided enrollment outcome, i.e., for each $i \in \{1, \ldots, n\}$, if $\mu^u(s_i) = c$ then $|\mu^d(c)| = q_c$, and
(3) there exists a student in the list whose unified enrollment school has a different type than the type of the unified enrollment school of another student in the list.

The third source of inefficiency is due to a *new interrupter*. In the 1-stage divided enrollment system, a student may apply to a school that he would not have applied to in the unified enrollment system, and interrupt the admission process of the school eventually not receiving an offer from that school but hurting some other students.

Say that a student $s \in S$ **suffers from a new interrupter** if there exists a list of students $(s_1, \ldots, s_n)$ and a student s' such that $s_1 = s$ and

(1) in the 1-stage divided enrollment system, each student in the list receives the unified enrollment school of the preceding student, i.e., for each $i \in \{2, \ldots, n\}$, $\mu^d(s_i) = \mu^u(s_{i-1})$, and student s_1 receives the unified enrollment school of student s_n, i.e., $\mu^d(s_1) = \mu^u(s_n)$,
(2) the school that each student receives in the unified enrollment system does not have any empty seats in the 1-stage divided enrolment outcome, i.e., for each $i \in \{1, \ldots, n\}$, if $\mu^u(s_i) = c$, then $|\mu^d(c)| = q_c$,
(3) the unified enrollment schools of all the students in the list are of the same type, and
(4) student s' is an interrupter at the unified enrollment school of a student in the list in the 1-stage divided enrollment system, although s' is not an interrupter at the same school in the unified enrollment system.

Theorem 9. *Each student whose 1-stage unified and 1-stage divided enrollment schools are different either suffers from waste, miscoordination, or a new interrupter.*

Proof of Theorem 9. Recall that S is the set of students who are matched to different schools at the truthful equilibrium of the 1-stage divided enrollment system and the truthful equilibrium of the 1-stage unified enrollment system. Suppose that $s \in S$, i.e., $\mu^u(s) \neq \mu^d(s)$, and also that student s does not suffer from waste. Since student s strictly prefers $\mu^u(s)$ to $\mu^d(s)$ and they are both individually rational for student s, $\mu^u(s)$ must be a school. Consider the set $S' \subseteq \mathcal{S}$ of students constructed by the following algorithm. Set $S' = \emptyset$ initially.

In Step 1, let S'_1 denote the set of students who are assigned to $\mu^u(s)$ at the 1-stage divided enrollment outcome, while they were assigned to a different school at the unified enrollment outcome, i.e., $S'_1 = \{s' \in S : \mu^d(s') = \mu^u(s)\}$. Update $S' = S'_1$.

In Step $n > 1$, let S'_n denote the set of students such that for each $s' \in S'_n$, there exists $s'' \in S'_{n-1}$ such that s' is assigned to $\mu^u(s'')$ at the at the 1-stage divided enrollment outcome, while s' is assigned to a different school at the unified enrollment outcome, i.e., $S'_n = \{s' \in S : \exists s'' \in S'_{n-1}$ such that $\mu^d(s') = \mu^u(s'')\}$. Update $S' = S'_n \cup S'_{n-1}$.

Since there are finitely many students, there exists a step M such that $S'_M = \emptyset$ and S' is constant after step M, where we stop the algorithm.

We claim that $s \in S'$. Since student s does not suffer from waste, for each $s' \in S' \cup \{s\}$, the capacity of $\mu^u(s')$ must be full at the 1-stage divided enrollment outcome. Let $C = \{c \in \mathcal{C} : \exists s' \in S' \cup \{s\}$ such that $\mu^u(s') = c\}$ be the set of unified enrollment schools of $S' \cup \{s\}$. For each $c \in C$, let $H(c)$ denote the set of students who are not assigned to school c at the unified enrollment but are assigned to school c at the 1-stage divided enrollment. Clearly, $| \cup_{c \in C} H(c)| \geq |S' \cup \{s\}|$. But also, $\cup_{c \in C} H(c) = S'$, implying that $s \in S'$.

Since $s \in S'$, there exists a list of students, including student s, satisfying the first two properties described in the definition of miscoordination. If there exists a student in the list whose unified enrollment school belongs to a different type than the unified enrollment school of another student in the list, then student s suffers from miscoordination. Otherwise, the list satisfies the properties 1, 2, and 3 in the definition of a new interrupter. Note that the unified enrollment schools of all the students in the list are of the same type.

Without loss of generality, suppose that for each student in the list, both his 1-stage unified and 1-stage divided enrollment schools are selective. Let t be the first step in SPDA for selective schools in the 1-stage divided enrollment system where a student in the list, say student s', is rejected by his unified enrollment school, say school c. Note that at Step t, there cannot be any other student from the list who also gets rejected by his unified enrollment school. Let student s^* be the lowest-priority student from among the students who are temporarily accepted by school c at Step t. Note that the student succeeding student s' receives the unified enrollment school of s' in the 1-stage divided enrollment system, and he therefore applies to school c after Step t and gets accepted by school c, implying that student s^* is an interrupter at school c since he must be rejected after Step t. Note that student s^* cannot be an interrupter at school c at the unified enrollment system, because otherwise student s^* prefers school c to his unified enrollment school while student s', who has a lower priority than student s^*, is assigned to school c in the unified enrollment system, which contradicts stability. Thus, student s suffers from a new interrupter. □

Appendix B. Omitted Proofs

In this section, we establish the omitted proofs. We start with the following result, which is helpful in proving Theorem 1.

Lemma 1. *The set of students who has applied to a school at or before Step t of the deferred acceptance algorithm in the unified enrollment system is a subset of those that have applied in the 1-stage divided enrollment system, for every t.*

Proof of Lemma 1. Without loss of generality, assume that $c \in \mathcal{C}^1$. Let $S^t(c)$ be the set of students who has applied to school c at or before Step t in the unified enrollment system. Similarly, let $\tilde{S}^t(c)$ be the set of students who has applied to school c at or before Step t in the divided enrollment system. We want to show that $S^t(c) \subseteq \tilde{S}^t(c)$ for every t. We prove this claim by mathematical induction on t.

When $t = 1$, each student applies to her top overall ranked school in the unified enrollment system. Therefore, the set of applicants in the unified enrollment system to school c is $S^1(c) = \{s|cP_sc' \text{ for every } c' \in \mathcal{C} \text{ such that } c' \neq c\}$. On the other hand, in the divided enrollment system, each student applies to the top-ranked school in every partition. That is, $\tilde{S}^1(c) = \{s|cP_sc' \text{ for every } c' \in \mathcal{C}^1 \text{ such that } c' \neq c\}$. Clearly, $S^1(c) \subseteq \tilde{S}^1(c)$.

Assume that the claim holds for every number smaller than t. We now show it for t. Let s be a student who has applied to school c at Step t in the unified enrollment system, that is, $s \in S^t(c) \backslash S^{t-1}(c)$. We claim that $s \in \tilde{S}^t(c)$. If school c is the highest-ranked school among $\mathcal{C}^1$ with respect to preference of student s, then $s \in \tilde{S}^1(c)$, which implies $s \in \tilde{S}^t(\mathrm{c})$. Otherwise, if school c is not the highest-ranked school, there exists a school $c' \in \mathcal{C}^1$ which is ranked right above c among schools in $\mathcal{C}^1$ with respect to the student's preference. Since $s \in S^t(c) \backslash S^{t-1}(c)$, school c' must have rejected student s at Step $t-1$ or before in the unified enrollment system, that is $s \in S^{t-1}(c')$ and $s \notin Ch_{c'}(S^{t-1}(c'); q_{c'})$. By the mathematical induction assumption, $s \in \tilde{S}^{t-1}(c')$ and by responsiveness $s \notin Ch_c(\tilde{S}^{t-1}(c'); q_{c'})$. Therefore, student s applies to school c at Step t or before in the divided enrollment system, so $s \in \tilde{S}^t(c)$.

We have shown that every student in $S^t(c) \setminus S^{t-1}(c)$, is also in $\tilde{S}^t(c)$. This and the mathematical induction hypothesis that $S^{t-1}(c) \subseteq \tilde{S}^{t-1}(c)$ imply $S^t(c) \subseteq \tilde{S}^t(c)$. □

Proof of Theorem 1. DA ends in a finite number of steps under both systems. Let μ be the matching produced by the unified enrollment system and $\tilde{\mu}_k$ be the matching produced by partition k in the divided enrollment system. Then the outcome for student s in the divided enrollment system is $\tilde{\mu}(s) \equiv \max_{P_s}\{\tilde{\mu}_1(s), \ldots, \tilde{\mu}_K(s)\}$. Assume, for contradiction, that $c \equiv \tilde{\mu}(s) P_s \mu(s)$.

Let t be bigger than the number of steps in DA under both systems and $S^t(c)$ be the set of students who has applied to school c at or before Step t in the unified enrollment system. Similarly, let $\tilde{S}^t(c)$ be the set of students who has applied to school c at or before Step t in the divided enrollment system. Since $c = \tilde{\mu}(s)$, $s \in Ch_c(\tilde{S}^t(c); q_c)$. Furthermore, since $cP_s\mu(s)$, student s must have been rejected by school c in the unified enrollment system. Therefore, $s \in S^t(c)$. But student s is not assigned to school c in the unified enrollment system, so $s \notin Ch_c(S^t(c); q_c)$. Lemma 1 shows that the set of applicants in the unified system is a subset of the set of applicants in the 1-stage divided enrollment system for any step, so $S^t(c) \subseteq \tilde{S}^t(c)$. This fact together with $s \notin Ch_c(S^t(c); q_c)$ and $s \in Ch_c(\tilde{S}^t(c); q_c)$ yield a contradiction because Ch_c is a responsive choice rule. □

The next lemma is used in the proof of Theorem 2.

Lemma 2. *Suppose that SPDA produces matching μ. When a set of students is removed and the capacities of their assigned schools in μ are reduced accordingly, the remaining students are weakly better off at the SPDA outcome of the new problem.*

The mathematical formulation of the result is as follows: For every market $(\mathcal{S}, (\mathcal{C}_k)_{k\in\{1,\ldots,K\}}, (R_s)_{s\in\mathcal{S}}, (q_c, \succeq_c)_{c\in\mathcal{C}})$, $S \subseteq \mathcal{S}$, and $s \in \mathcal{S} \setminus S$, the outcome of SPDA under $(\mathcal{S} \setminus S, (\mathcal{C}_k)_{k\in\{1,\ldots,K\}}, (R_s)_{s\in\mathcal{S}\setminus S}, (q_c - |\mu(c) \cap S|, \succeq_c)_{c\in\mathcal{C}})$ is weakly better with respect to R_s than the outcome of SPDA under $(\mathcal{S}, (\mathcal{C}_k)_{k\in\{1,\ldots,K\}}, (R_s)_{s\in\mathcal{S}}, (q_c, \succeq_c)_{c\in\mathcal{C}})$.

Proof of Lemma 2. Let μ' be the matching obtained from μ by removing the set of students S together with their assigned seats at μ. Note that matching μ' includes only the students in $\mathcal{S} \setminus S$. First we show that μ' is a stable matching in the new market $(\mathcal{S} \setminus S, (\mathcal{C}_k)_{k\in\{1,\ldots,K\}}, (R_s)_{s\in\mathcal{S}\setminus S}, (q_c - |\mu(c) \cap S|, \succ_c)_{c\in\mathcal{C}})$. Individual rationality for students holds since μ is individually rational in $(\mathcal{S}, \mathcal{C}, (R_s)_{s\in\mathcal{S}}, (q_c, \succ_c)_{c\in\mathcal{C}})$. For any school c, individual rationality in $(\mathcal{S}, (\mathcal{C}_k)_{k\in\{1,\ldots,K\}}, (R_s)_{s\in\mathcal{S}}, (q_c, \succ_c)_{c\in\mathcal{C}})$ implies that $Ch_c(\mu(c); q_c) = \mu(c)$. In μ', the set of students matched to school c is $\mu(c) \setminus S$ and the capacity of c is $q_c - |\mu(c) \cap S|$. Since Ch_c is the responsive choice rule we get $Ch_c(\mu(c) \setminus S; q_c - |\mu(c) \cap S|) = \mu(c) \setminus S$ which shows that the μ' is individually rational for school c. Any blocking pair (s, c) for μ' in the new problem $(\mathcal{S} \setminus S, (\mathcal{C}_k)_{k\in\{1,\ldots,K\}}, (R_s)_{s\in\mathcal{S}\setminus S}, (q_c - |\mu(c) \cap S|, \succ_c)_{c\in\mathcal{C}})$ would also be a blocking pair in the original problem $(\mathcal{S}, (\mathcal{C}_k)_{k\in\{1,\ldots,K\}}, (R_s)_{s\in\mathcal{S}}, (q_c, \succ_c)_{c\in\mathcal{C}})$ because of responsiveness, so blocking pairs cannot exist. Therefore, μ' is stable in the new problem.

Since μ' is stable in the new problem, the outcome of SPDA for the new problem is weakly better for the remaining students since SPDA produces the student-optimal stable matching (Gale and Shapley, 1962). The conclusion follows. □

Proof of Theorem 2. Lemma 2 shows that removing students and their allotments in SPDA weakly improves the outcome for the remaining students in SPDA. Lemma 2 and the result that removing a student from a matching market weakly improves the rest of the students under SPDA (Kelso and Crawford, 1982; Chambers and Yenmez, 2013) yield the first statement. The reason is as follows. Since students submit their preferences truthfully in both stages of the divided enrollment system for all types of schools and since some students may leave by choosing one of the schools that has admitted them, the outcome of SPDA improves weakly for all the remaining students in all types of schools. Therefore, for each type of school, a student gets a weakly better school.

The second statement follows from the first statement as no student can get a better outcome by switching from rejecting all offers in the first stage to accepting one of the schools at the first stage.

□

Proof of Theorem 3. Consider any Nash equilibrium outcome of the 2-stage divided enrollment system, say μ. We first show that there is no blocking pair. Suppose, for contradiction, that there exist students s, s' and school c such that $\mu(s') = c$, $c \; P_s \; \mu(s)$, and $s \succ_c s'$. Without loss of generality, suppose that c is a selective school. That is, call schools that have the same type as school c selective schools.

Case 1: Both students s and s' accept their best offers. In this case, the outcome in the market for selective schools in the first stage has a blocking pair, which contradicts with the stability of the SPDA.

Case 2: Student s' accepts his best offer and student s rejects all offers at the first stage. Then, the selective school that made student s an offer in the first stage, if any, must be worse than school c, since otherwise student s would have a profitable deviation such that he instead accepts the best offer. But then, the outcome in the market for selective schools in the first stage again has a blocking pair, which contradicts with the stability of the SPDA.

Case 3: Student s accepts his best offer and student s' rejects all offers at the first stage. Consider a unilateral deviation of student s such that he instead rejects all offers.

Now, in the second round in the submarket for selective schools, there will be one additional student, s, and at most one more seat for a selective school. Consider the submarket without the additional seat, but only with the additional student s. First note that there cannot be an interrupter in this market. This is not immediately implied by our efficient capacity-priority structure assumption, since we are looking at a submarket where the capacities of the schools may be smaller than their original capacities. However, by the argument in Case 2, for each student who receives a seat in Stage 2, and for each school that he prefers to his assigned school in Stage 2, all the students who receive that school in the first stage must have a higher priority. Now consider the market obtained from the submarket by adding the students who received seats in the first round and the seats that were allocated in the first round, with the new preference profile where the additional students find acceptable only the school that they were assigned in the first round. Clearly, any student who is an interrupter in the submarket will still be an interrupter in this new market where the schools have their original capacities, which cannot happen. As a result, there cannot be an interrupter in the submarket.

Next, we will show that, since initially student s' receives school c and student s has higher priority than student s' at school c, student s gets either school c or something better in the submarket after his one-shot deviation. Suppose, for contradiction, that student s gets a school that is worse than school c. For any step of SPDA at which student s is tentatively accepted by a school that he prefers to school c, no other student should be rejected from that school at the same step, since there is no interrupter. Also, for any step of SPDA at which student s is tentatively accepted by school c, no other student should be rejected from school c at the same step. Now, consider the step at which student s is rejected by school c. Note that until student s gets rejected by school c, SPDA if student s had not deviated to participate in the second round and if he deviated to participate in the second round run exactly the same. Also note that, since student s gets a selective school that is worse than school c and the allocation of selective schools must be stable, student s' must be rejected from school c at some step. If student s' is tentatively accepted at c in the same step student s gets rejected by school c, then student s is an interrupter at school c, which is a contradiction. Suppose that student s' is not one of the tentatively accepted students at school c in the step that student s gets rejected by school c. Then, there exists a student s'' who has lower priority than student s' at school c and gets tentatively accepted by school c in the step student s gets rejected by school c, since in SPDA when student s does not deviate to participate in the second round, student s' applies to school c at a later step and gets tentatively accepted. But then, student s'' gets rejected by school c at a later step, by the latest at the first step student s' applies to school c. Hence, student s'' is an interrupter at school c, which is a contradiction.

Thus, student s gets either school c or something better after his unilateral deviation in the submarket without the added seat. By resource monotonicity of SPDA (Kelso and Crawford, 1982; Chambers and Yenmez, 2013), we conclude that student s gets either school c or something better after his unilateral deviation, which is a profitable unilateral deviation. Thus, we get a contradiction.

Case 4: Students s and s' reject all offers at the first stage. Since students submit their true preferences at the second stage and SPDA is stable, student s' cannot be matched with school c while student s does not receive an offer from school c or a better selective school with respect to her preference ranking R_s because student s has a higher priority than student s' at school c. This is a contradiction.

In all possible cases, we have established a contradiction. Hence, matching μ has no blocking pairs. In addition, it must be individually rational since in equilibrium a student does not accept an unacceptable school and a school never chooses unacceptable students. Yet this does not mean that matching μ is stable because it can have waste: a student may prefer an empty seat to her assigned seat. Consider the reduced market obtained by decreasing the capacities of the schools (by the amount of empty seats). Then, matching μ is stable at this new problem. Let matching μ^* be the student-optimal stable matching at this problem. Note that each student is weakly better off at matching μ^* compared to matching μ. Also, by resource monotonicity of the SPDA, when the school capacities are increased to the initial levels, each student must be weakly better off, which is the unified enrollment outcome, compared to μ^*, and also compared to μ. Hence, there is no student at μ who is better off compared to the outcome of the unified enrollment system. □

Proof of Theorem 4. Some of the arguments in this proof are the same as in the proof of Theorem 3. For completeness, we repeat these arguments.

Consider any SPNE outcome of the 2-stage divided enrollment system, say μ. We first show that there is no blocking pair. Suppose, for contradiction, that there exist students s, s' and school c such that $\mu(s') = c$, $c \; P_s \; \mu(s)$, and $s \succ_c s'$. Suppose also that student s' is the lowest-priority student at school c in matching μ. Without loss of generality, suppose that c is a selective school. That is, call schools that have the same type as school c selective schools.

Case 1: Both students s and s' accept their best offers and get matched at the first stage. Let $\tilde{R}_s$ denote the preference relation that student s reports for the selective schools. Since SPDA is used, the selective school assigned to student s is better than school c with respect to preference relation $\tilde{R}_s$.

Consider a one-shot deviation of student s at the preference revelation decision node where she instead reports another preference relation over selective schools, say $\tilde{R}'_s$, which is obtained from $\tilde{R}_s$ just by moving school c to the top.

Consider the subcase when student s receives school c as a selective school. In the following subgame, his strategy either recommends she accept school c or recommends she reject all matched schools since she is not matched with a school better than school c at the first step, and moreover she receives a school at least as good as school c at the end of the game, since her strategy induces a Nash equilibrium in every subgame. In any case, there is a profitable one-shot deviation, which is a contradiction.

Consider the other subcase when student s does not receive school c as a selective school at the first step. Since there cannot be any interrupter, in any step of SPDA where student

s is tentatively accepted by school c, no other student is rejected by c at the same step. And, as a result, SPDA produces the same matching as when he reports $\tilde{R}_s$. Therefore, student s' is matched with school c, which contradicts the stability of SPDA.

Case 2: Student s' accepts his best offer and student s rejects all offers at the first stage. Then, the selective school that made student s an offer in the first stage, if any, must be worse than school c, since otherwise student s would have a profitable one-shot deviation at an accept-reject decision node where he instead accepts the best offer. But then, due to the same arguments as in Case 1, student s has a profitable one-shot deviation at the preference revelation decision node by moving school c to the top of his preference ranking over selective schools.

Case 3: Student s accepts his best offer and student s' rejects all offers at the first stage. Consider a one-shot deviation of student s at the accept-reject decision node where he instead rejects all offers.

Now in the second round in the submarket for selective schools, there will be one additional student, s, and at most one more seat for a selective school. Consider the submarket without the additional seat, but only with the additional student s. First note that there cannot be an interrupter in this market. This is not immediately implied by our efficient capacity-priority structure assumption, since we are looking at a submarket where the capacities of the schools may be smaller than their original capacities. However, by the argument in Case 2, for each student who receives a seat in Stage 2, and for each school that he prefers to his assigned school in Stage 2, all the students who receive that school in the first stage must have a higher priority. Now, consider the market obtained from the submarket by adding the students who received seats in the first round and the seats that were allocated in the first round, with the new preference profile where the additional students find acceptable only the school that they were assigned in the first round. Clearly, any student who is an interrupter in the submarket will still be an interrupter in this new market where the schools have their original capacities, which cannot happen. As a result, there cannot be an interrupter in the submarket.

Next, we will show that, since initially student s' receives school c and student s has higher priority than student s' at school c, student s gets either school c or something better in the submarket after his one-shot deviation. Suppose, for contradiction, that student s gets a school that is worse than school c. For any step of SPDA at which student s is tentatively accepted by a school that he prefers to school c, no other student should be rejected from that school at the same step, since there is no interrupter. Also, for any step of SPDA at which student s is tentatively accepted by school c, no other student should be rejected from c at the same step. Now, consider the step at which s is rejected by c. Note that until student s gets rejected by school c, SPDA if student s had not deviated to participate in the second round and if he deviated to participate in the second round run exactly the same. Also note that, since student s gets a selective school that is worse than school c and the allocation of selective schools must be stable, student s' must be rejected from school c at some step. If student s' is tentatively accepted at school c in the same step student s gets rejected by school c, then student s is an interrupter at school c, which is a contradiction. Suppose that student s' is not one of the tentatively accepted students at school c in the step that student s gets rejected by school c. Then, there exists a student s'' who has lower priority than student s' at school c and gets tentatively accepted by school c at the step student s gets rejected by school c, since in SPDA when student s had not deviated to participate in the second round, student s' applies to school c at a later step and gets tentatively accepted. But then, student s'' gets rejected by school c at a later step, by the latest at the first step student s' applies to school c. Hence, student s'' is an interrupter at school c, which is a contradiction.

Thus, student s gets either school c or something better after his one-shot deviation in the submarket without the added seat. By resource monotonicity of SPDA (Kelso and Crawford, 1982; Chambers and Yenmez, 2013), we conclude that student s gets either school c or something better after his one-shot deviation, which is a profitable one-shot deviation. Thus, we get a contradiction.

Case 4: Students s and s' accept their best offers at the second stage. Since students submit their true preferences at the second stage and SPDA is stable, student s' cannot be matched with school c while student s does not receive an offer from school c or a better selective school with respect to her preference ranking R_s because student s has a higher priority than student s' at school c. This is a contradiction.

In all possible cases, we have established a contradiction. Hence, matching μ has no blocking pairs. In addition, it must be individually rational since in equilibrium a student does not accept an unacceptable school and a school never chooses unacceptable students. Yet this does not mean that matching μ is stable because it can have waste: a student may prefer an empty seat to her assigned seat. Consider the reduced market obtained by decreasing the capacities of the schools (by the amount of empty seats). Then, matching μ is stable at this new problem. Let matching μ^* be the student-optimal stable matching at this problem. Note that each student is weakly better off at matching μ^* compared to matching μ. Also, by resource monotonicity of the SPDA, when the school capacities are increased to the initial levels, each student must be weakly better off, which is the unified enrollment outcome, compared to μ^* and also compared to μ. Hence, there is no student at μ who is better off compared to the outcome of the unified enrollment system. □

Proof of Theorem 5. The first statement for the divided system is from Theorem 4, which also implies the statement for the unified system. Thus, both systems have an equilibrium in weakly-dominant strategies where students reject all offers after the first stage. For each system, this equilibrium outcome is the outcome of the corresponding 1-stage enrollment system outcome in which each agent submits their preferences truthfully. Therefore, the last statement follows from Theorem 1. □

Proof of Theorem 6. We will prove the statement for the 2-stage divided enrollment system. The case for the unified enrollment system follows by the same arguments.

Let s be a student, σ be the strategy profile where all students, including s, reject their offers, and σ' be any other strategy profile where student s still rejects his offers but some other students accept their offers. Then, by Lemma 2 and by the resource monotonicity of SPDA (Kelso and Crawford, 1982; Chambers and Yenmez, 2013), student s receives weakly better offers for all types of schools in the second stage under strategy profile σ' than under strategy profile σ. Therefore, he weakly prefers the outcome under strategy profile σ' to the outcome under strategy profile σ.

Now consider any equilibrium strategy profile σ^*. Since σ^* is an equilibrium, at the strategy profile σ^{**} where student s rejects his offers and the other students follow their σ^* strategies, student s is weakly worse off (note that, if student s rejects his offers in σ^*, then $\sigma^* = \sigma^{**}$). Moreover, by the above arguments, student s weakly prefers the outcome under σ^{**} to the outcome when all students reject all offers. Thus, student s weakly prefers the outcome under σ^* to the outcome when all students reject all offers. Hence, rejecting all offers is the worst equilibrium for all students. □

Proof of Theorem 7. When the capacity-priority structure is efficient for the unified enrollment system, SPDA satisfies consistency (Ergin, 2002). As a result, when some students leave the market with their allotments, SPDA produces the same matching for the remaining students. Therefore, each student is indifferent between accepting her offer after the first stage and rejecting it regardless of what other students do. More formally, for each student, accepting the offer and rejecting it are payoff equivalent. This means that every strategy profile is a Nash equilibrium. Furthermore, all the equilibria produce the truthful equilibrium outcome of SPDA.

The last statement that the equilibrium outcome in the 2-stage unified enrollment system is weakly better than all the equilibria of the 2-stage divided enrollment system follows from Theorem 3. □

Proof of Theorem 8. Since there exists a SPNE, for each profile of preferences $\overline{R}$ reported by students in the first stage, the following subgame has a Nash equilibrium. Let $\sigma^*(\overline{R})$ be a Nash equilibrium of the subgame following $\overline{R}$, where the associated strategy of each student s is $\sigma^*_s(\overline{R})$.

Consider the following strategy profile. In the first stage, each student reports acceptable only his 1-stage unified enrollment school. This is the school that he is matched with when everyone reports truthfully in the 1-stage unified enrollment system.

After the preference-profile realization where all the students report only their 1-stage unified enrollment schools acceptable, each student accepts his offer (which is his 1-stage unified enrollment school).

After any preference profile realization where all students except for one report acceptable only their 1-stage unified enrollment schools, each student accepts his offer if and only if he weakly prefers his offer to his 1-stage unified enrollment school.

After any other preference profile realization $\overline{R}$ (where at least two students do not submit the preferences stated above), each student s plays the strategy $\sigma_s^*(\overline{R})$.

Consider any student s. Consider his preference revelation decision node. Suppose that, by a one-shot deviation, he changes his preference report. By the stability of the unified enrollment system, the offer he receives is either from his unified enrollment school or from a worse school. If he receives an offer from his unified enrollment school, his strategy recommends he accept it, which is not a profitable one-shot deviation. If he receives an offer from a worse school that has an empty seat at the unified enrollment outcome, then all the other students still receive offers from their 1-stage unified enrollment schools and accept their offers, while student s rejects his offer and in the second stage gets matched to his 1-stage unified enrollment school, which is again not a profitable one-shot deviation. Suppose that student s receives an offer from a worse school that is fully allocated in the 1-stage unified enrollment outcome. Then, one of the other students, say student s', receives an offer from his outside option in the first stage, and all the students other than s and s' receive offers from their 1-stage unified enrollment schools. Then, all the students other than s and s' accept their offers in the first stage, and in the second stage students s and s' participate in the market consisting of the unassigned seats at the 1-stage unified enrollment outcome together with one seat of the 1-stage unified enrollment school of s and one seat of the 1-stage unified enrollment school of s'. Since in this market the best school for s is his 1-stage unified enrollment school, we again do not have a profitable one-shot deviation.

Consider the acceptance-rejection decision of student s after the preference-profile realization where all the students report acceptable only their 1-stage unified enrollment schools. Note that each student gets only one offer, which is from their 1-stage unified enrollment school, and any seat not allocated in the first stage is not preferable by student s to his unified enrollment school. Thus, if student s deviates by rejecting the offer, in the second stage, he still receives his 1-stage unified enrollment school, which is not a profitable deviation.

Consider an acceptance-rejection decision of student s after all the students except for one student, say s^*, report acceptable only their 1-stage unified enrollment schools. If $s = s^*$, then the offer of student s is not strictly preferred to his 1-stage unified enrollment outcome, and clearly he cannot be better off by deviating from his strategy since he reports only his 1-stage unified enrollment outcome acceptable. Suppose that $s \neq s^*$. The nontrivial subcase here is when student s^* receives an offer from the 1-stage unified enrollment school of another student s^{**}, student s^{**} receives an offer from his outside option, and student s receives an offer from his 1-stage unified enrollment school. If student s, by a one-shot deviation, instead rejects his offer, then in the second stage students s, s^*, and s^{**} participate in the market consisting of the unassigned seats at the 1-stage unified enrollment outcome together with the one seat at the 1-stage unified enrollment school of student s, the one seat at the 1-stage unified enrollment school of student s^*, and the one seat at the 1-stage unified enrollment school of student s^{**}. Suppose that this is a profitable one-shot deviation. Then, without loss of generality, suppose that in the second stage student s receives the 1-stage unified enrollment school of student s^*, which he strictly prefers to his 1-stage unified enrollment school. By stability of the 1-stage unified enrollment system, student s^* has higher priority than student s at his 1-stage unified enrollment school. Then, student s^* must also be matched to a more preferred school in the second stage. If student s^* is matched to the 1-stage unified enrollment school of student s, then the 1-stage unified enrollment outcome is not Pareto efficient, contradicting the efficient capacity-priority structure assumption. If student s^* is matched to the 1-stage unified enrollment school of student s^{**}, then student s^{**} must be matched to the 1-stage unified enrollment school of student s that he strictly prefers to his 1-stage unified enrollment school, again similarly contradicting the efficient capacity-priority structure assumption.

At any other acceptance-rejection decision node, following a preference profile $\overline{R}$, student s clearly does not have a profitable one-shot deviation since $\sigma^*(\overline{R})$ is a Nash equilibrium of the following subgame. $\square$

Appendix C. Examples

Example 1. (*When students are strategic only in acceptance-rejection decisions, the 2-stage divided enrollment system may have an equilibrium that produces the same outcome as the 1-stage unified enrollment system.*) Let $\mathcal{S} = \{s_1, s_2\}$ and $\mathcal{C} = \{c_1, c_2, c_3\}$. Each school has capacity one. Suppose that school c_1 is a selective school and schools c_2 and c_3 are nonselective schools. Let the student preference profile R and the school priority profile $\succeq$ be depicted as below.

In the unified enrollment system, SPDA matches student s_1 with school c_1 and student s_2 with school c_2 when students are truthful. Now let us analyze the 2-stage divided enrollment

R_{s_1}	R_{s_2}	$\succeq_{c_1}$	$\succeq_{c_2}$	$\succeq_{c_3}$
c_1	c_2	s_1	s_1	s_2
c_2	c_1	s_2	s_2	s_1
c_3	c_3	c_1	c_2	c_3
s_1	s_2			

system. In the first stage, student s_1 gets offers from schools c_1 and c_2, and student s_2 gets an offer from school c_3. The normal form representation of the game is below.

		s_2: Accept best	s_2: Reject all
s_1	Accept best	(c_1, c_3)	(c_1, c_2)
	Reject all	(c_1, c_3)	(c_1, c_3)

Note that, for student s_1, accepting the best offer and rejecting all offers are payoff equivalent. Also note that there is a Nash equilibrium which achieves the unified enrollment outcome: student s_1 accepts the best offer, student s_2 rejects all offers, and they both receive their top-ranked schools.

Example 2. (*When students are strategic only in acceptance-rejection decisions, the 2-stage divided enrollment system equilibria outcomes may be Pareto dominated by the unified enrollment outcome.*) Let $\mathcal{S} = \{s_1, s_2\}$ and $\mathcal{C} = \{c_1, c_2\}$. Each school has capacity one. Suppose that school c_1 is a selective school and school c_2 is a nonselective school. Let the student preference profile and school priority profile be depicted as below.

R_{s_1}	R_{s_2}	$\succeq_{c_1}$	$\succeq_{c_2}$
c_1	c_2	s_2	s_1
c_2	c_1	s_1	s_2
s_1	s_2	c_1	c_2

In the unified enrollment system, student s_1 gets matched with school c_1 and student s_2 gets matched with school c_2. Now let us analyze the divided enrollment system. In the first stage, student s_1 gets an offer from c_2 and student s_2 gets an offer from c_1. The normal form representation of the game is below.

		s_2: Accept best	s_2: Reject all
s_1	Accept best	(c_2, c_1)	(c_2, c_1)
	Reject all	(c_2, c_1)	(c_2, c_1)

Note that the game has a unique Nash equilibrium outcome, which is Pareto dominated by the unified enrollment outcome in which both students get their top-ranked schools.

Example 3. (*When students are strategic only in acceptance-rejection decisions, the 2-stage divided enrollment system equilibrium outcome may Pareto dominate the 1-stage unified enrollment outcome.*) Let $\mathcal{S} = \{s_1, s_2, s_3\}$ and $\mathcal{C} = \{c_1, c_2, c_3\}$. Each school has capacity one. Suppose that all the schools are nonselective schools. Let the student preference profile R and the school priority profile $\succeq$ be depicted as below.

R_{s_1}	R_{s_2}	R_{s_3}	$\succeq_{c_1}$	$\succeq_{c_2}$	$\succeq_{c_3}$
c_1	c_1	c_2	s_3	s_2	s_1
c_3	c_2	c_1	s_1	s_3	s_2
c_2	c_3	c_3	s_2	s_1	s_3
s_1	s_2	s_3	c_1	c_2	c_3

In the first stage, student s_1 gets an offer from school c_3, student s_2 gets an offer from school c_2, and student s_3 gets an offer from school c_1. Consider the strategy profile where student s_1 accepts his best offer and the other two students reject all offers. The outcome of this strategy profile is that student s_1 gets school c_3, student s_2 gets school c_1, and student s_3 gets school c_2. Since students s_2 and s_3 get their best schools, they do not have any profitable unilateral deviation. If student s_1 instead rejects, he still receives school c_3, so he also does not have a profitable unilateral deviation. Therefore, we have a Nash equilibrium outcome that Pareto dominates the unified enrollment outcome in which student s_1 gets school c_3, student s_2 gets school c_2, and student s_3 gets school c_1.

Example 4. (*The 2-stage divided enrollment system may have no pure-strategy SPNE, even in a market with an efficient capacity-priority structure for divided enrollment.*) Let $\mathcal{S} = \{s_1, s_2, s_3\}$ and $\mathcal{C} = \{c_1, c_2, c_3, c_4\}$. Each school has capacity one. Suppose that schools c_1, c_2, and c_3 are nonselective schools and school c_4 is a selective school. Let the student preference profile R and the school priority profile $\succeq$ be depicted as below.

R_{s_1}	R_{s_2}	R_{s_3}	$\succeq_{c_1}$	$\succeq_{c_2}$	$\succeq_{c_3}$	$\succeq_{c_4}$
c_4	c_1	c_2	s_3	s_3	s_3	s_2
c_1	c_2	c_3	s_2	s_2	s_2	s_1
c_3	c_4	c_4	s_1	s_1	s_1	s_3
s_1	s_2	s_3	c_1	c_2	c_3	c_4
c_2	c_3	c_1				

First note that the market has an efficient capacity-priority structure for divided enrollment (although it is not efficient for unified enrollment) since all the selective schools have the same priority ordering over students and all the nonselective schools have the same priority ordering over students. Consider the subgame that follows after student s_1 reports school c_1 as the only acceptable nonselective school and does not report any selective school as acceptable; student s_2 reports school c_2 as the only acceptable nonselective school and does not report any selective school as acceptable; and student s_3 reports school c_3 as the only acceptable nonselective school and does not report any selective school as acceptable. Note that student s_1 gets an offer from school c_1, student s_2 gets an offer from school c_2, and student s_3 gets an offer from school c_3. We show that this subgame does not have a pure strategy Nash equilibrium.

Here are the eight possible pure-strategy profiles:

All students accept: Student s_1 gets school c_1, student s_2 gets school c_2, and student s_3 gets school c_3. If student s_1 unilaterally deviates to reject, he gets school c_4 and becomes better off.

Only student s_1 rejects: Student s_1 gets school c_4, student s_2 gets school c_2, and student s_3 gets school c_3. If student s_2 unilaterally deviates to reject, he gets school c_1 and becomes better off.

Only student s_2 rejects: Student s_1 gets school c_1, student s_2 gets school c_2, and student s_3 gets school c_3. If student s_3 unilaterally deviates to reject, he gets school c_2 and becomes better off.

Only student s_3 rejects: Student s_1 gets school c_1, student s_2 gets school c_2, and student s_3 gets school c_3. If student s_1 unilaterally deviates to reject, he gets school c_4 and becomes better off.

Only students s_1 and s_2 reject: Student s_1 gets his outside option s_1, student s_2 gets school c_1, and student s_3 gets school c_3. If student s_1 unilaterally deviates to accept, he gets school c_1 and becomes better off.

Only students s_1 and s_3 reject: Student s_1 gets school c_4, student s_2 gets school c_2, and student s_3 gets school c_3. If student s_2 unilaterally deviates to reject, he gets school c_1 and becomes better off.

Only students s_2 and s_3 reject: Student s_1 gets school c_1, student s_2 gets school c_4, and student s_3 gets school c_2. If student s_2 unilaterally deviates to accept, he gets school c_2 and becomes better off.

All students reject: Student s_1 gets school c_3, student s_2 gets school c_1, and student s_3 gets school c_2. If student s_1 unilaterally deviates to accept, he gets school c_1 and becomes better off.

Since there is a subgame without a pure-strategy Nash equilibrium, there is no pure-strategy SPNE.

Example 5. (*The 2-stage divided enrollment system equilibrium outcome may Pareto dominate the 1-stage unified enrollment outcome.*)

Consider Example 3 again. The unified enrollment outcome is that student s_1 gets school c_3, student s_2 gets school c_2, and student s_3 gets school c_1. Consider the following strategy profile. Student s_1 reports only school c_3 as acceptable in the first stage, and accepts the best offer if and only if it is strictly preferred to his unified enrollment outcome school c_3. Each other student reports truthfully in the first stage and at each possible preference profile realization, and he accepts the best offer if and only if it is strictly preferred to his unified enrollment outcome.

Consider student s_1. Consider his preference revelation decision node. By changing his preference report, he either causes an outcome where student s_2 gets school c_2 and student s_3 gets school c_1 or an outcome where student s_2 gets school c_1 and student s_3 gets school c_2. In the former case, all students reject their offers, and student s_1 in the end receives school c_3. In the latter case, only student s_1 rejects his offer, and in the end he still receives school c_3. So there is no one-shot profitable deviation at his preference revelation decision node. Consider any of his acceptance-rejection decision nodes. If he is offered school c_1, he can clearly not be strictly better off by rejecting it. If he is offered school c_3 and he rejects, in the worse case he receives school c_3 in the second stage, so again there is no one-shot profitable deviation. If he receives school c_2 and rejects it, then he receives a school at the second stage that cannot be worse than school c_2. Again, there is no profitable one-stage deviation.

Consider student s_2. Consider his preference revelation decision node. Clearly, there is no one-shot profitable deviation since he gets his best school. Consider any of his acceptance-rejection decision nodes. If he is offered school c_1, he can clearly not be better off by rejecting the offer. If he is offered school c_2 and he rejects, in the worse case he receives school c_2 in the second stage, so there is no one-shot profitable deviation either. If he is offered school c_3 in the first stage, by rejecting this offer he gets a school at the second stage that cannot be worse than school c_3. Again there is no one-stage deviation.

The case for student s_3 is symmetrical to the case for student s_2. Thus, student s_3 does not have a one-stage deviation either.

Example 6. (*When students are strategic only in acceptance-rejection decisions, the 2-stage divided enrollment system may not have a best equilibrium for all students.*) Let $\mathcal{S} = \{s_1, s_2, s_3\}$ and $\mathcal{C} = \{c_1, c_2, c_3, c_4\}$. Each school has capacity one. Suppose that schools c_1, c_2 are nonselective schools and schools c_3 and c_4 are selective schools. Let the student preference profile R and the school priority profile $\succeq$ be depicted as below.

Note that in the first stage, student s_1 gets offers from schools c_1 and c_3, student s_2 gets offers from schools c_2 and c_4, and student s_3 gets no offer. Whether student s_1 accepts or

R_{s_1}	R_{s_2}	R_{s_3}	$\succeq_{c_1}$	$\succeq_{c_2}$	$\succeq_{c_3}$	$\succeq_{c_4}$
c_1	c_3	c_2	s_1	s_2	s_1	s_2
c_2	c_4	s_3	s_2	s_1	s_2	s_1
c_3	c_1	c_1	s_3	s_3	s_3	s_3
s_1	s_2	c_3	s_4	s_4	s_4	s_4
c_4	c_2	c_4	c_1	c_2	c_3	c_4

rejects his offer, he gets his best school c_1, since he has top priority at school c_1. So there are two types of equilibria depending on whether student s_1 accepts or rejects his offer.

Suppose that student s_1 accepts his offer. Then the unique best response of student s_2 is to reject his offer, since independent from what student s_3 chooses, student s_2 can receive school c_3 in the second stage. In this case, student s_3 is indifferent between accepting or rejecting, since in any case he receives his outside option s_3. Therefore, there is a unique equilibrium outcome where student s_1 accepts his offer: (c_1, c_3, s_3).

Suppose that student s_1 rejects his offer. Then student s_2 is indifferent between accepting or rejecting, since independent from what student s_3 chooses, student s_2 gets school c_4 in any case. If student s_2 accepts, the unique best response of student s_3 is to reject and receive school c_2 in the second stage. If student s_2 rejects, student s_3 is indifferent between accepting or rejecting and in any case he receives his outside option s_3. So, there are two possible equilibrium outcomes where student s_1 rejects his offer: (c_1, c_4, s_3) or (c_1, c_4, c_2).

Hence, there are three possible equilibrium outcomes: (c_1, c_3, s_3), (c_1, c_4, s_3), and (c_1, c_4, c_2). Clearly, there is no student-optimal equilibrium.