Top Trading Cycles, Consistency, and Acyclic Priorities for House Allocation with Existing Tenants*

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Abstract

We study the house allocation with existing tenants model (introduced by Abdulkadiroğlu and Sönmez, 1999) and consider rules that allocate houses based on priorities. We introduce a new acyclicity requirement for the underlying priority structure which is based on the acyclicity conditions by Ergin (2002) and Kesten (2006) for house allocation with quotas and without existing tenants. We show that for house allocation *with* existing tenants a top trading cycles rules is *consistent* if and only if its underlying priority structure satisfies our acyclicity condition. Moreover, even if no priority structure is *a priori* given, we show that a rule is a top trading cycles rule based on ownership-adapted acyclic priorities if and only if it satisfies *Pareto-optimality*, *individual-rationality*, *strategy-proofness*, *reallocation-proofness*, and *consistency*.

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1 Introduction

Motivated by real life on-campus housing practices, Abdulkadiroğlu and Sönmez (1999) introduced *house allocation problems with existing tenants*: A finite set of houses has to be allocated to a finite set of agents without using monetary transfers. Each agent is either an existing tenant who occupies a house or a new applicant, and each house is either occupied or vacant. Furthermore, each agent has strict preferences over all houses and the so-called null house (or outside option). An outcome for a house allocation problem with existing tenants is a matching that assigns to each agent either a real house or the null house, such that no real house is assigned to more than one agent. A rule selects a matching for each house allocation problem with existing tenants.

A house allocation problem with existing tenants reduces to a *housing market* (Shapley and Scarf, 1974) if there are no new applicants and no vacant houses, i.e., all agents are existing tenants and all houses are occupied (we discuss some related literature on housing markets in Appendix C.1). A house allocation problem with existing tenants reduces to a *house allocation problem* (Hylland and Zeckhauser, 1979) if there are no existing tenants and no occupied houses, i.e., all agents are new applicants and all houses are vacant (we discuss some related literature on house allocation problems in Appendix C.2).

An important question that emerged in both the literature on housing markets as well as the house allocation problem is the characterization of rules that allocate houses in a *Pareto-optimal*[1] and *strategy-proof*[2] way. For housing markets, Roth and Postlewaite (1977, Theorem 2') showed that the core of a housing market[3] is unique and it is the outcome of the top trading cycles (TTC) algorithm. Roth (1982) showed that the core / TTC rule is *strategy-proof*, and Bird (1984) showed that it is also *group strategy-proof*.[4] Ma (1994, Theorem 1) characterized the core / TTC rule of a housing market by *Pareto-optimality*, *individual-rationality (for tenants)*,[5] and *strategy-proofness*; see also Sönmez (1999, Corollary 3) and Svensson (1999, Theorem 2).

For house allocation problems, Pápai (2000) introduced *hierarchical exchange rules*: Hierarchical exchange rules extend the way TTC rules work by specifying ownership rights for the houses in an iterative hierarchial manner and by allowing for associated iterative trades. Pápai (2000) showed that a rule for house allocation problems satisfies *Pareto-optimality*, *group strategy-proofness*, and *reallocation-proofness*[6] if and only if it is a hierarchical ex-

[1]A rule is *Pareto-optimal* if the matching chosen by the rule is such that there is no other matching that makes some agents better off without hurting the others.

[2]A rule is *strategy-proof* if no agent can ever benefit by misrepresenting his preferences unilaterally.

[3]A matching for a housing market is in the *core* (or *core stable*) if no subset of agents exists such that some of them strictly benefit by reallocating their occupied houses among themselves, without hurting other agents in the group.

[4]A rule is *group strategy-proof* if no group of agents can ever benefit by misrepresenting their preferences.

[5]A rule for housing markets satisfies *individual-rationality (for tenants)* if no agent is assigned a house that is worse for him than his occupied house.

[6]A rule for house allocation problems is *reallocation-proof* if there do not exist two agents who gain by first misreporting their preferences and then swapping their assigned houses, such that neither of the two

change rule. The analysis of Pápai (2000) was extended by Pycia and Ünver (2017), who provided a full characterization of the class of *Pareto-optimal* and *group strategy-proof* rules without relying on *reallocation-proofness*.

For house allocation problems, if *consistency*[7] is considered in addition to *Pareto-optimality* and *(group) strategy-proofness*, then rules based on acyclic priorities become focal: For house allocation with quotas problems, which reduce to house allocation problems when the quota of each house is one, Ergin (2002) and Kesten (2006) studied rules that allocate houses based on *acyclic priorities*.[8] Ergin (2002, Theorem 1) showed that for the agents-proposing deferred acceptance rule (Gale and Shapley, 1962) based on a priority structure π, denoted by DA^{π}, the following are equivalent: DA^{π} is *Pareto-optimal*, DA^{π} is *group strategy-proof*, DA^{π} is *consistent*, and π is (Ergin) acyclic. Kesten (2006, Theorems 1 and 2) showed that the TTC rule based on the priority structure is the agents-proposing deferred acceptance rule based on the priority structure if and only if the priority structure is (Kesten) acyclic, and the TTC rule based on the priority structure is *consistent* if and only if the priority structure is (Kesten) acyclic.

In the work of Ergin (2002) and Kesten (2006), the priority structure is exogenously given. The class of TTC rules based on priority structures is a subclass of the hierarchical exchange rules studied by Pápai (2000). Hence, under *group strategy-proofness and reallocation-proofness*, priorities can endogenously arise. This is also the case under *consistency* (Ehlers and Klaus, 2006): a rule is an *efficient priority* rule if it adapts to an Ergin acyclic priority structure[9] and the assignment of houses to agents are determined by the agents-proposing deferred acceptance rule. Ehlers and Klaus (2006, Proposition 2 and Theorem 1) characterized efficient priority rules for house allocation problems by *Pareto-optimality*, *group strategy-proofness*, and *reallocation-consistency*.[10]

Motivated by these previous works, we extend the analysis of *Pareto-optimal*, *(group) strategy-proof*, and *consistent* rules to the more general model of house allocation with existing tenants. We extend the notion of (Ergin / Kesten) acyclicity to this model and show that a TTC rule based on ownership-adapted priorities[11] is *consistent*[12] if and only if the priority

agents can change his assignment by misreporting alone.

[7]A rule for house allocation problems is *consistent* if the following holds: suppose that after houses are allocated according to the rule, some agents leave the house allocation problem with their assigned houses. Then, if the remaining agents were to allocate the remaining houses according to the rule, each of them would receive the same house.

[8]Ergin (2002) and Kesten (2006) use different notions of acyclicity that, however, coincide for house allocation problems. We discuss house allocation with quotas problems and Ergin and Kesten acyclicity in detail in Appendix C.3).

[9]For house allocation problems, a rule *adapts to a priority structure* if no agent has a higher priority for the house assigned to another agent that he would prefer to his assigned house.

[10]A rule for house allocation problems is *reallocation-consistent* if the following holds: suppose that after houses are allocated according to the rule some agents are removed from the problem with their assigned houses, then, if these removed agents were to allocate their assigned houses (the removed houses) among themselves according to the rule, each of them would receive the same house.

[11]Priorities are *ownership-adapted* if each tenant has top priority at his occupied house.

[12]A rule for house allocation problems with existing tenants is *consistent* if the following holds: suppose

structure is *acyclic* (Theorem 1). As the second main result (Theorem 2), we characterize TTC rules based on ownership-adapted acyclic priorities by *Pareto-optimality*, *individual-rationality*, *strategy-proofness*, *reallocation-proofness*, and *consistency*. This result generalizes the characterizations of the core / TTC rule for housing markets by Ma (1994) (see Corollary 1) and of efficient priority rules for house allocation problems by Ehlers and Klaus (2006) (see Corollary 3).

One important precursor of our study is the work of Sönmez and Ünver (2010) on the so called YRMH-IGYT (you request my house - I get your turn) rules introduced by Abdulkadiroğlu and Sönmez (1999) for house allocation with existing tenants. Sönmez and Ünver (2010, Theorem 1) showed that a rule for house allocation problems with existing tenants satisfies *Pareto-optimality*, *individual-rationality*, *strategy-proofness*, *weak neutrality*,[13] and *consistency* if and only if it is a YRMH-IGYT rule. Their work relates to ours as follows. The class of TTC rules based on ownership-adapted acyclic priorities that we study is a superset of the class of YRMH-IGYT rules. Moreover, a TTC rule based on ownership-adapted acyclic priorities can be interpreted as a two-step rule where the first rule is an *almost YRMH-IGYT rule* and the second rule is an *efficient priority rule* of Ehlers and Klaus (2006) (see Proposition 3).

The paper is organized as follows. In Section 2 we introduce the house allocation with existing tenants model and basic properties of rules. In Section 3 we introduce priority structures and TTC rules. Furthermore, we show that a TTC rule based on ownership-adapted priorities satisfies *Pareto-optimality*, *individual-rationality*, *strategy-proofness*, and *reallocation-proofness* (Proposition 1) and that a TTC rule based on ownership-adapted priorities is *consistent* if and only if the priority structure is *acyclic* (Theorem 1). In Section 4 we state and prove our characterization of TTC rules that are based on ownership-adapted acyclic priorities by *Pareto-optimality*, *individual-rationality*, *strategy-proofness*, *reallocation-proofness*, and *consistency* (Theorem 2). Section 5 concludes by showing how our results imply previous results by Ma (1994) (Corollary 1), Kesten (2006) (Corollary 2), and Ehlers and Klaus (2006) (Corollary 3) and by interpreting a TTC rule based on ownership-adapted acyclic priorities as a two-step rule where the first rule is an almost YRMH-IGYT rule and the second rule is an efficient priority rule (Proposition 3).

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that after houses are allocated according to the rule we remove some agents with their assignments and some unassigned houses from the problem in a way that the reduced problem contains the occupied houses of all remaining existing tenants, i.e., we never remove an occupied house without its existing tenant. Then, if the remaining agents were to allocate the remaining houses according to the rule, each of them would receive the same house.

[13]A rule for house allocation problems with existing tenants is *weakly neutral* if it is independent of the names of the vacant houses.

2 House Allocation with Existing Tenants and Basic Properties

We mostly follow Sönmez and Ünver (2010) in this section.

Let $\mathcal{I}$ be a finite **set of potential agents** and $\mathcal{H}$ be a finite **set of potential houses**. Without loss of generality we assume that $|\mathcal{I}| \geq 3$ and $|\mathcal{H}| \geq 2$. Let h_0 denote the **null house**. We interpret the null house as the outside option of an agent if he does not receive any house. We fix a global **ownership structure** $h : \mathcal{I} \to \mathcal{H} \cup \{h_0\}$. An agent $i \in \mathcal{I}$ is either an **existing tenant**, i.e., he already occupies a house $h(i) \in \mathcal{H}$, or a **new applicant**, i.e., $h(i) = h_0$. No two agents can occupy the same house in $\mathcal{H}$, i.e., for each $i, j \in \mathcal{I}$ with $h(i) = h(j) \neq h_0$ we have $i = j$. Let $\mathcal{I}_E$ denote the set of **potential existing tenants** and $\mathcal{I}_N$ the set of **potential new applicants**; the set of potential agents $\mathcal{I}$ is partitioned into the sets $\mathcal{I}_E$ and $\mathcal{I}_N$.[14]

For each agent $i \in \mathcal{I}$ and set of houses $H \subseteq \mathcal{H}$, let $\mathcal{R}(i, H)$ denote the set of all linear orders over $H \cup \{h_0\}$.[15] For each agent $i \in \mathcal{I}$, we interpret $R_i \in \mathcal{R}(i, H)$ as agent i's (strict) **preferences** over houses in H and the null house h_0; e.g., for $h, h', h'' \in H$, $[R_i : h\, P_i\, h'\, P_i\, h_0\, P_i\, h''\, P_i \ldots]$ means that agent i would first like to have house h, then to have h', and then i would prefer to have the null house h_0 rather than house h'', etc. An agent $i \in \mathcal{I}$ finds a house $h \in \mathcal{H}$ **acceptable** if $h\, P_i\, h_0$. We assume that every existing tenant $i \in \mathcal{I}_E$ finds the house that he already occupies acceptable, i.e., $h(i)\, P_i\, h_0$. Let $\mathcal{R}(I, H)$ denote the **set of all preference profiles** over $H \cup \{h_0\}$ for agents in I, i.e., $\mathcal{R}(I, H) = \prod_{i \in I} \mathcal{R}(i, H)$.

Given $R \in \mathcal{R}(I, H)$ and $\widetilde{I} \subseteq I$, let $R_{\widetilde{I}}$ denote the preference profile $(R_i)_{i \in \widetilde{I}}$; it is the **restriction of $\boldsymbol{R}$ to the set of agents $\boldsymbol{\widetilde{I}}$**. We also use the notation $R_{-\widetilde{I}} = R_{I \setminus \widetilde{I}}$ and $R_{-i} = R_{I \setminus \{i\}}$.

A **house allocation problem with existing tenants** is a list (I, H, R), where

(i) $I \subseteq \mathcal{I}$ is a finite set of agents,

(ii) $H \subseteq \mathcal{H}$ is a finite set of houses such that for each existing tenant $i \in I \cap \mathcal{I}_E$, $h(i) \in H$, and

(iii) $R = (R_i)_{i \in I} \in \mathcal{R}(I, H)$ is a preference profile.

Note that by (ii), if an existing tenant is present, then so is the house he occupies.

[14] In contrast to Sönmez and Ünver (2010) we do not require that there exists at least one house that is not occupied by a potential existing tenant, i.e., we do not require $|\mathcal{H}| > |\mathcal{I}_E|$.

[15] A linear order over $H \cup \{h_0\}$ is a binary relation $\widehat{R}$ that is *antisymmetric* (for each $h, h' \in H \cup \{h_0\}$, if $h\, \widehat{R}\, h'$ and $h'\, \widehat{R}\, h$, then $h = h'$), *transitive* (for each $h, h', h'' \in H \cup \{h_0\}$, if $h\, \widehat{R}\, h'$ and $h'\, \widehat{R}\, h''$, then $h\, \widehat{R}\, h''$), and *complete* (for each $h, h' \in H \cup \{h_0\}$, $h\, \widehat{R}\, h'$ or $h'\, \widehat{R}\, h$). By $\widehat{P}$ we denote the asymmetric part of $\widehat{R}$. Hence, given $h, h' \in H \cup \{h_0\}$, $h\, \widehat{P}\, h'$ means that h is *strictly preferred* to h'; $h\, \widehat{R}\, h'$ means that $h\, \widehat{P}\, h'$ or $h = h'$ and that h is *weakly preferred* to h'.

Let $\mathfrak{D}$ denote the domain of all house allocation problems with existing tenants (house allocation problems with existing tenants were introduced by Abdulkadiroğlu and Sönmez, 1999).

Given a problem $(I, H, R) \in \mathfrak{D}$, $I_E = \mathcal{I}_E \cap I$ denotes the **set of existing tenants**, $I_N = I \setminus I_E = \mathcal{I}_N \cap I$ denotes the **set of new applicants**, $H_O = \{h(i)\}_{i \in I_E}$ denotes the **set of occupied houses**, and $H_V = H \setminus H_O$ denotes the **set of vacant houses**.

A house allocation problem with existing tenants is called **a house allocation problem with only new applicants** when there are no existing tenants and hence no occupied houses, i.e., $I_E = \emptyset$ and $H_O = \emptyset$. Let $\mathfrak{D}^N$ denote the domain of all house allocation problems with only new applicants. A house allocation problem with only new applicants is traditionally called a **house allocation problem** (house allocation problems were first analyzed in Hylland and Zeckhauser, 1979).

A house allocation problem with existing tenants is called **a house allocation problem with only existing tenants** when there are no new applicants and no vacant houses, i.e., $I_N = \emptyset$ and $H_V = \emptyset$. Let $\mathfrak{D}^E$ denote the domain of all house allocation problems with only existing tenants. A house allocation problem with only existing tenants is traditionally called a **housing market** (housing markets were first introduced by Shapley and Scarf, 1974).

We denote a **generic domain** by $\mathcal{D}$ ($\mathcal{D} \subseteq \mathfrak{D}$). Note that we will not restrict agents' preferences throughout this article, domain restrictions only focus on restrictions of the ownership structure.

A **matching** for a house allocation problem with existing tenants $(I, H, R) \in \mathcal{D}$ is a function $\mu : I \to H \cup \{h_0\}$ such that no two agents are assigned to the same house in H, i.e., for each $h \in H$, $|\mu^{-1}(h)| \leq 1$ (the null house can be assigned to more than one agent). Given a matching μ for $(I, H, R) \in \mathcal{D}$ and an agent $i \in I$, $\mu(i) \in (H \cup \{h_0\})$ denotes the house agent i is matched to under μ and is referred to as the **allotment of agent *i***. For each agent $i \in I$ and matchings μ, μ', we let $\mu \; R_i \; \mu'$ if and only if $\mu(i) \; R_i \; \mu'(i)$, i.e., agents only care about their own allotments but not how the remaining houses are allocated.

A **rule** ϕ on $\mathcal{D}$ is a function that associates with each problem $(I, H, R) \in \mathcal{D}$ a matching $\phi(I, H, R)$. Given a problem $(I, H, R) \in \mathcal{D}$, an agent $i \in I$, and a rule ϕ, $\phi_i(I, H, R)$ denotes the allotment of agent i at matching $\phi(I, H, R)$. For a group of agents $I' \subseteq I$ we define $\phi_{I'}(I, H, R) := \bigcup_{i \in I'} \phi_i(I, H, R)$.

The first property of a rule we introduce is the well-known condition of *Pareto-optimality*.

Definition 1 (**Pareto-Optimality**). A matching μ is *Pareto-optimal* for problem $(I, H, R) \in \mathcal{D}$ if there is no other matching μ' for problem (I, H, R) such that for each agent $i \in I$, $\mu' \; R_i \; \mu$ and for some $j \in I$, $\mu' \; P_j \; \mu$. A rule ϕ on $\mathcal{D}$ is *Pareto-optimal* if it only assigns *Pareto-optimal* matchings.

Next, we introduce voluntary participation conditions based on the idea that no existing tenant can be forced to be assigned a house that is worse than the house he already occupies and no agent can be forced to be matched to a house that is unacceptable to him.

Definition 2 (**Individual-Rationality**). A matching μ is *individually-rational for tenants* for problem $(I, H, R) \in \mathcal{D}$ if for each tenant $i \in I_E$, $\mu(i)\, R_i\, h(i)$. A matching μ is *individually-rational* for problem $(I, H, R) \in \mathcal{D}$ if for each agent $i \in I$, $\mu(i)\; R_i\; h(i)$. A rule ϕ on $\mathcal{D}$ is *individually-rational (for tenants)* if it only assigns *individually-rational (for tenants)* matchings.

Note that *Pareto-optimality* and *individual-rationality for tenants* together imply *individual-rationality* (since every existing tenant finds his occupied house acceptable and if a new applicant receives an unacceptable allotment, then we can make him better off by assigning the null house to him without making any other agent worse off). For simplicity we use the stronger notion of *individual-rationality* but we could use the weaker version of *individual-rationality for tenants* throughout.

The well-known non-manipulability property *strategy-proofness* requires that no agent can ever benefit from misrepresenting his preferences.

Definition 3 (**Strategy-Proofness**). A rule ϕ on $\mathcal{D}$ is *strategy-proof* if for each problem $(I, H, R) \in \mathcal{D}$, each agent $i \in I$, and each preference relation $\widetilde{R}_i \in \mathcal{R}(i, H)$,

$$\phi_i(I, H, R)\; R_i\; \phi_i(I, H, (\widetilde{R}_i, R_{-i})).$$

The next property was introduced by Pápai (2000) to exclude joint preference manipulation by two individuals who plan to swap objects ex post under the condition that the collusion changed both their allotments and is self enforcing in the sense that neither agent changes his allotment in case he misreports while the other agents reports the truth.

Definition 4 (**Reallocation-Proofness**). A rule ϕ on $\mathcal{D}$ is *reallocation-proof* if for each problem $(I, H, R) \in \mathcal{D}$ and each pair of agents $i, j \in I$, there exist no preference relations $\widetilde{R}_i \in \mathcal{R}(i, H)$ and $\widetilde{R}_j \in \mathcal{R}(j, H)$ such that

$$\phi_j(I, H, (\widetilde{R}_i, \widetilde{R}_j, R_{-\{i,j\}}))\; R_i\; \phi_i(I, H, R),$$

$$\phi_i(I, H, (\widetilde{R}_i, \widetilde{R}_j, R_{-\{i,j\}}))\; P_j\; \phi_j(I, H, R),$$

and

$$\phi_k(I, H, R) = \phi_k(I, H, (\widetilde{R}_k, R_{-k})) \neq \phi_k(I, H, (\widetilde{R}_i, \widetilde{R}_j, R_{-\{i,j\}})) \text{ for } k = i, j.$$

Next, we formulate a *consistency* notion for house allocation with tenants (as introduced by Sönmez and Ünver, 2010): if some agents leave a house allocation problem with tenants with their allotments and possibly some unassigned houses are removed, as long as no tenant is left behind while his occupied house is removed, the rule should allocate the remaining houses among the agents who did not leave in the same way as in the original house allocation problem with tenants. For house allocation problems with only new applicants (on $\mathfrak{D}^N$), *consistency* for house allocation problem with tenants implies *reallocation-consistency*[10] as introduced by Ehlers and Klaus (2006) as well as the standard *consistency*[7] property (e.g., Ehlers and Klaus, 2007; Ergin, 2000, 2002). We introduce some notation before defining *consistency*.

For each agent $i \in I$, preference relation $R_i \in \mathcal{R}(i,H)$, and set of houses $\widehat{H} \subseteq H$, let $R_i^{\widehat{H}} \in \mathcal{R}(i,\widehat{H})$ denote the **restriction of $\boldsymbol{R_i}$ to houses in $\widehat{\boldsymbol{H}} \cup \{\boldsymbol{h_0}\}$**, i.e., for each $h, \widehat{h} \in \widehat{H} \cup \{h_0\}$, $h\, R_i^{\widehat{H}}\, \widehat{h}$ if and only if $h\, R_i\, \widehat{h}$. Given fixed $H \subseteq \mathcal{H}$, $\widehat{H} \subsetneq H$, and $R_i \in \mathcal{R}(i,H)$, we denote the restriction of R_i to houses in $(H \setminus \widehat{H}) \cup \{h_0\}$ by $R_i^{-\widehat{H}}$, i.e., $R_i^{H\setminus\widehat{H}} = R_i^{-\widehat{H}}$. For each $\widehat{I} \subseteq I$ and $\widehat{H} \subseteq H$, let $R_{\widehat{I}}^{\widehat{H}} = (R_i^{\widehat{H}})_{i\in\widehat{I}}$ denote the **restriction of preference profile $\boldsymbol{R}$ to agents in $\widehat{\boldsymbol{I}}$ and houses in $\widehat{\boldsymbol{H}} \cup \{\boldsymbol{h_0}\}$**.

Given a house allocation problem with tenants $(I,H,R) \in \mathfrak{D}$, $\widehat{I} \subseteq I$, and $\widehat{H} \subseteq H$, $(\widehat{I},\widehat{H},R_{\widehat{I}}^{\widehat{H}})$ is the **restriction of $(\boldsymbol{I},\boldsymbol{H},\boldsymbol{R})$ to agents in $\widehat{\boldsymbol{I}}$ and houses in $\widehat{\boldsymbol{H}} \cup \{\boldsymbol{h_0}\}$**. The restricted problem $(\widehat{I},\widehat{H},R_{\widehat{I}}^{\widehat{H}})$ is a **reduced problem**, i.e., $(\widehat{I},\widehat{H},R_{\widehat{I}}^{\widehat{H}}) \in \mathfrak{D}$, if the occupied houses of existing tenants in $\widehat{I}$ is the set of occupied houses in $\widehat{H}$, that is $\widehat{H}_O = \cup_{i\in\widehat{I}_E} h(i)$.

Definition 5 (Consistency). A rule ϕ on $\mathcal{D}$ is *consistent* if for each problem $(I,H,R) \in \mathcal{D}$ and each removal of a set of agents $\widehat{I} \subsetneq I$ together with their allotments under ϕ, $\widehat{H} = \phi_{\widehat{I}}(I,H,R)$, and some unassigned houses $\widetilde{H} \subseteq H$ that results in a reduced problem $(I \setminus \widehat{I}, H \setminus (\widehat{H} \cup \widetilde{H}), R_{-\widehat{I}}^{-(\widehat{H}\cup\widetilde{H})}) \in \mathcal{D}$, it follows that for each agent $i \in (I \setminus \widehat{I})$,

$$\phi_i(I,H,R) = \phi_i(I \setminus \widehat{I}, H \setminus (\widehat{H} \cup \widetilde{H}), R_{-\widehat{I}}^{-(\widehat{H}\cup\widetilde{H})}).$$

Note that *consistency* only imposes a restriction on a rule defined on $\mathcal{D}$ for reduced problems that are in $\mathcal{D}$.

We call a domain $\mathcal{D} \subseteq \mathfrak{D}$ **closed (under reduction)** if each reduced problem is in $\mathcal{D}$ again, i.e., for each $(I,H,R) \in \mathcal{D}$ and each $\widehat{I} \subsetneq I$, if $(I \setminus \widehat{I}, H \setminus (\widehat{H} \cup \widetilde{H}), R_{-\widehat{I}}^{-(\widehat{H}\cup\widetilde{H})})$ is a reduced problem, then $(I \setminus \widehat{I}, H \setminus (\widehat{H} \cup \widetilde{H}), R_{-\widehat{I}}^{-(\widehat{H}\cup\widetilde{H})}) \in \mathcal{D}$.

The domain of house allocation problems with existing tenants $\mathfrak{D}$, the domain of house allocation problems with only new applicants $\mathfrak{D}^N$, and the domain of house allocation problems with only existing tenants $\mathfrak{D}^E$ are each closed domains.

For house allocation problems with only new applicants, a well-known property that is implied by *consistency*, called *non-bossiness*, requires that whenever a change in an agent's preference relation does not bring about a change in his allotment, it does not bring about a change in anybody's allotment (Satterthwaite and Sonnenschein, 1981).

Definition 6 (Non-Bossiness). A rule ϕ on $\mathcal{D}$ is *non-bossy* if for each problem $(I,H,R) \in \mathcal{D}$, each agent $i \in I$, and each preference relation $\widetilde{R}_i \in \mathcal{R}(i,H)$, if $\phi_i(I,H,R) = \phi_i(I,H,(\widetilde{R}_i,R_{-i}))$, then $\phi(I,H,R) = \phi(I,H,(\widetilde{R}_i,R_{-i}))$.

Remark 1 (**Consistency does not imply Non-Bossiness**). For house allocation problems with only new applicants, we can easily see why *consistency* implies *non-bossiness*: If a new applicant unilaterally changes his preferences such that he receives the same allotment, then, since this allotment is a vacant house, in each of the two problems he can leave with his allotment and two reduced problems in the domain of house allocation problems result. These two problems are identical and hence have to have the same matching. Thus, by *consistency*, also the matchings in the two original problems have to have been identical.

For house allocation problems with tenants, *consistency* does *not* imply *non-bossiness* anymore. The reason is that if an agent i now unilaterally changes his preferences such that he receives the same allotment, and this allotment is an occupied house, when leaving with only the occupied house, its tenant is left behind and hence the resulting reduced problems are not in the domain of house allocation with tenants problems and hence, *consistency* has no bite. One then could try to remove the smallest set of tenants together with agent i such that house allocation with tenants problems result. However, the set of removed tenants or their allotments need not be the same and different reduced house allocation with tenants problems might result. Again, we cannot conclude that the matchings in the two original problems have to have been identical. □

Strategy-proofness and *non-bossiness* together imply *group strategy-proofness* (see, e.g., Pápai, 2000).

Definition 7 (**Group Strategy-Proofness**). A rule ϕ on $\mathcal{D}$ is *group strategy-proof* if for each problem $(I, H, R) \in \mathcal{D}$, there is no group of agents $\widetilde{I} \subseteq I$ and no preference profile $\widetilde{R} \in \mathcal{R}(\widetilde{I}, H)$, such that for all $i \in \widetilde{I}$, $\phi_i(I, H, (\widetilde{R}, R_{-\widetilde{I}})\ R_i\ \phi_i(I, H, R)$ and for some $j \in \widetilde{I}$, $\phi_j(I, H, (\widetilde{R}, R_{-\widetilde{I}})\ P_j\ \phi_j(I, H, R)$.

A recent survey (Thomson, 2016) discusses many other logical relationships of *non-bossiness* with well-known normative or strategic properties.

3 Priority Structures and Top Trading Cycles Rules

Let $\Pi^{\mathcal{I}}$ denote all one-to-one functions from $\{1, \ldots, |\mathcal{I}|\}$ to $\mathcal{I}$. For each $h \in \mathcal{H}$, $\pi^h \in \Pi^{\mathcal{I}}$ denotes the **priority ordering for house *h***. Here agent $\pi^h(1)$ has the top priority at house h, agent $\pi^h(2)$ has the second priority at h, and so on. By a slight abuse of notation we will also denote the inverse function $(\pi^h)^{-1}$ by π^h such that for an agent $i \in \mathcal{I}$, $\pi^h(i) \in \{1, \ldots, |\mathcal{I}|\}$ denotes his rank in the priority. For a set of agents $I \subseteq \mathcal{I}$ we define the **restriction of $\boldsymbol{\pi^h}$ to $\boldsymbol{I}$** to be a one-to-one function $\pi_I^h \in \Pi^I$ such that $\pi_I^h(i) < \pi_I^h(j)$ if and only if $\pi^h(i) < \pi^h(j)$. A **priority structure** is a list $\pi \equiv \{\pi^h \mid \pi^h \in \Pi^{\mathcal{I}}\}_{h \in \mathcal{H}}$ of priority orderings, one for each house in $\mathcal{H}$. For a set of agents $I \subseteq \mathcal{I}$, $\pi_I \equiv \{\pi_I^h\}_{h \in \mathcal{H}}$ denotes a **restricted priority structure**. Note that $\pi_{\mathcal{I}} = \pi$.

We say that a priority structure π is **adapted to the ownership structure** if each existing tenant has top priority at his own house, i.e., for each $i \in \mathcal{I}_E$, $\pi^{h(i)}(1) = i$.

Any priority structure π can be adapted to the ownership structure by moving every existing tenant to the top of the priority ordering of his own house without changing the ordering of other agents. Formally, for each priority structure $\pi \equiv \{\pi^h \mid \pi^h \in \Pi^{\mathcal{I}}\}_{h\in\mathcal{H}}$ the ownership-adapted priority structure $\widehat{\pi} \equiv \{\widehat{\pi}^h \mid \widehat{\pi}^h \in \Pi^{\mathcal{I}}\}_{h\in\mathcal{H}}$ is such that

(a) for each vacant house $h \in \mathcal{H}_V$, $\widehat{\pi}^h := \pi^h$ and

(b) for each occupied house $h(i) \in \mathcal{H}_O$,

$\widehat{\pi}^{h(i)}(1) = i$ and

for each $j, k \in \mathcal{I} \setminus \{i\}$, $\widehat{\pi}^{h(i)}(j) < \widehat{\pi}^{h(i)}(k)$ if and only if $\pi^{h(i)}(j) < \pi^{h(i)}(k)$.

Given a problem (I, H, R), a priority structure π, and a matching μ, we say that μ **violates the priority of agent $\boldsymbol{i \in I}$ for house $\boldsymbol{h \in H}$** if there exists an agent $j \in I$ such that $\mu(j) = h$, $\pi^h(i) < \pi^h(j)$, and $h \, P_i \, \mu(i)$, i.e., agent i has higher priority for house h than agent j but j receives h and i justifiably envies j. A rule ϕ on $\mathcal{D}$ **adapts to a priority structure $\boldsymbol{\pi}$** if for each problem $(I, H, R) \in \mathcal{D}$, $\phi(I, H, R)$ does not violate the priority of any agent for any house.

For more general house allocation problems where each house can have multiple identical copies, the house allocation with quotas model (also known as school choice model), Ergin (2002) and Kesten (2006) introduced acyclicity conditions for priority structures that coincide for house allocation problems with only new applicants (on $\mathfrak{D}^N$); see Lemma 6 in Appendix C.3. We extend their acyclicity notion to house allocation problems with tenants.

Definition 8 (Acyclicity). For a set of agents $I \subseteq \mathcal{I}$ and a restricted priority structure π_I that is adapted to the ownership structure, π_I is *acyclic* if for agents $i, j, k \in I$ and houses $h, h' \in \mathcal{H}$ such that h' is not owned by any of the three agents, i.e., $h' \notin \{h(i), h(j), h(k)\}$,

$$\pi^h(i) < \pi^h(j) < \pi^h(k) \text{ implies } [\pi^{h'}(i) < \pi^{h'}(k) \text{ or } \pi^{h'}(j) < \pi^{h'}(k)].$$

First, as already mentioned above, for house allocation problem with only new applicants (on $\mathfrak{D}^N$) our definition of acyclicity coincides with Ergin and Kesten acyclicity.[16] Second, for house allocation problem with only existing tenants (on $\mathfrak{D}^E$), every priority structure that is adapted to the ownership structure is acyclic (since there are no vacant houses).

The following is an example of an acyclic priority structure.

Example 1 (An Acyclic Priority Structure). Table 1 gives an example of an acyclic priority structure π for a market with three existing tenants a, b, c, five new applicants d, e, f, g, i and eight houses $h(a), h(b), h(c), h_1, h_2, h_3, h_4, h_5$.

[16] In general, if a priority structure is Kesten acyclic, then it is also Ergin acyclic (note that in the more general house allocation with quotas model additional "scarcity conditions" are used to define Ergin and Kesten cycles).

n	$\pi^{h(a)}(n)$	$\pi^{h(b)}(n)$	$\pi^{h(c)}(n)$	$\pi^{h_1}(n)$	$\pi^{h_2}(n)$	$\pi^{h_3}(n)$	$\pi^{h_4}(n)$	$\pi^{h_5}(n)$
1	a	b	c	d	d	d	d	d
2	d	d	d	b	b	b	b	b
3	b	c	b	c	c	c	c	c
4	c	a	e	e	a	a	e	e
5	e	e	a	a	e	e	a	a
6	f	g	f	f	f	f	g	f
7	g	f	g	g	g	g	f	g
8	i	i	i	i	i	i	i	i

Table 1: An acyclic priority structure.

$\square$

For each house allocation problem with tenants $(I, H, R) \in \mathcal{D}$ and each priority structure π we define the **top trading cycles (TTC) rule based on priority structure $\boldsymbol{\pi}$** recursively using Gale's **top trading cycles (TTC) algorithm** (Shapley and Scarf, 1974, attributed the TTC algorithm that finds a core allocation in housing markets to David Gale):

Input. A house allocation problem with tenants $(I, H, R) \in \mathfrak{D}$ and a priority structure π.

Step 1. Let $I_1 := I$ and $H_1 := H$. We construct a (directed) graph with the set of nodes $I_1 \cup H_1 \cup \{h_0\}$.

For each agent $i \in I_1$ we add a directed edge to his most preferred house in $H_1 \cup \{h_0\}$. For each directed edge (i, h) ($i \in I_1$ and $h \in H_1$) we say that agent i points to house h. For each house $h \in H_1$ we add a directed edge to the highest ranked agent in I_1 in its priority ordering, i.e., to $\pi^h_{I_1}(1)$. For the null house we add a directed edge to each agent in I_1.

A **trading cycle** is a directed cycle in the graph. Given the finite number of nodes, at least one trading cycle exists for the graph. We assign to each agent in a trading cycle the house he points to and remove all trading cycle agents and houses. We define I_2 to be the set of remaining agents and H_2 to be the set of remaining houses and, if $I_2 \neq \emptyset$, we continue with Step 2. Otherwise we stop.

In general at Step t we have the following:

Step *t*. We construct a (directed) graph with the set of nodes $I_t \cup H_t \cup \{h_0\}$ where $I_t \subseteq I$ is the set of agents that remain after Step $t-1$ and $H_t \subseteq H$ is the set of houses that remain after Step $t-1$.

For each agent $i \in I_t$ we add a directed edge to his most preferred house in $H_t \cup \{h_0\}$. For each house $h \in H_t$ we add a directed edge to the highest ranked agent in I_t in its priority ordering, i.e., to $\pi^h_{I_t}(1)$. For the null house we add a directed edge to each agent in I_t.

At least one trading cycle exists for the graph and we assign to each agent in a trading cycle the house he points to and remove all trading cycle agents and houses. We define I_{t+1} to be the set of remaining agents and H_{t+1} to be the set of remaining houses and, if $I_{t+1} \neq \emptyset$, we continue with Step $t+1$. Otherwise we stop.

Output. The TTC algorithm terminates when all agents in I are assigned a house in $H \cup \{h_0\}$ (it takes at most $|I|$ steps). We denote the house in $H \cup \{h_0\}$ that agent $i \in I$ obtains in the TTC algorithm by $\varphi_i^\pi(I, H, R)$.

The **TTC rule (on $\boldsymbol{\mathcal{D}}$) based on priority structure $\boldsymbol{\pi}$, $\boldsymbol{\varphi^\pi}$**, associates with each problem $(I, H, R) \in \mathcal{D}$ the matching determined by the TTC algorithm.

The next example illustrates the TTC rule with the acyclic priority structure of Example 1.

Example 2 (A TTC Rule). Let $I = \{a, b, c, d, e, f, g, i\}$ and $H = \{h(a), h(b), h(c), h_1, h_2, h_3, h_4, h_5\}$. Agents $I_E = \{a, b, c\}$ are tenants. Consider the preference profile $R \in \mathcal{R}(I, H)$ defined as in Table 2.

R_a	R_b	R_c	R_d	R_e	R_f	R_g	R_i
h_1	h_1	$h(b)$	$h(c)$	h_1	h_1	h_1	h_1
h_2	$h(b)$	$h(c)$	h_0	$h(c)$	h_2	$h(a)$	h_2
$h(a)$				h_5	h_4	h_0	h_3
				h_0	h_0		h_0

Table 2: A preference profile.

We consider the TTC assignment with priority structure π given at Table 1 for problem (I, H, R). Table 3 gives the set of agents and houses present in the steps of the TTC and the top trading cycles that form in each step.

t	I_t	$H_t \cup \{h_0\}$	trading cycles in Step t
1	$\{a, b, c, d, e, f, g, i\}$	$\{h(a), h(b), h(c), h_1, h_2, h_3, h_4, h_5, h_0\}$	$[d, h(c), c, h(b), b, h_1]$
2	$\{a, e, f, g, i\}$	$\{h(a), h_2, h_3, h_4, h_5, h_0\}$	$[a, h_2]$, $[e, h_5]$
3	$\{f, g, i\}$	$\{h(a), h_3, h_4, h_0\}$	$[f, h_4, g, h(a)]$
4	$\{i\}$	$\{h_3, h_0\}$	$[i, h_3]$

Table 3: Steps of the TTC algorithm.

In the final assignment $\mu = \varphi^\pi(I, H, R)$, we have $\mu(a) = h_2$, $\mu(b) = h_1$, $\mu(c) = h(b)$, $\mu(d) = h(c)$, $\mu(e) = h_5$, $\mu(f) = h_4$, $\mu(g) = h(a)$, and $\mu(i) = h_3$. $\square$

All TTC rules based on ownership-adapted priority structures are *Pareto-optimal*, *individually-rational*, *group strategy-proof*, and *reallocation-proof*.

Proposition 1 ($\boldsymbol{\varphi^\pi}$: Pareto-Optimality, Individual-Rationality, Group Strategy-Proofness, Reallocation-Proofness). *For each priority structure π that is adapted to the ownership structure, the TTC rule on domain $\mathcal{D}$ based on π, φ^π, satisfies* Pareto-optimality, individual-rationality, group strategy-proofness, *and* reallocation-proofness.

Proof. *Individual-rationality* of φ^π follows from the facts that π is adapted to the ownership structure and no agent points to a house that is unacceptable for him at any step of the TTC algorithm. *Pareto-optimality* and *group strategy-proofness* follow from the well-known fact that TTC rules satisfy these properties. In particular, each TTC rule based on a priority structure is a "hierarchical exchange rule" and therefore satisfies *Pareto-optimality*, *group strategy-proofness*, and *reallocation-proofness* (see Pápai, 2000). $\square$

The following example demonstrates that not all TTC rules based on ownership-adapted priority structures are *consistent*.

Example 3 (Cyclic Priorities and Violation of Consistency). We provide a simple example with a cyclic priority structure π and show that the TTC rule based on π violates *consistency*.

Let (I, H, R) be a problem where $I = \{i, j, k\}$ with $i \in I_E$, $H = \{h, h(i)\}$ with $h \in H_V$, and preferences $R = (R_i, R_j, R_k)$ are such that

- $R_i : h \; P_i \; h(i) \; P_i \; h_0$,
- $R_j : h(i) \; P_j \; h_0 \; P_j \; h$, and
- $R_k : h(i) \; P_k \; h_0 \; P_k \; h$.

We consider a priority structure π restricted to I and H where $\pi^{h(i)}(i) < \pi^{h(i)}(j) < \pi^{h(i)}(k)$ and $\pi^h(k) < \pi^h(i) < \pi^h(j)$. Note that the priority structure π is cyclic.

The TTC rule based on the priority structure π for problem (I, H, R) results in allotments $\varphi^\pi_i(I, H, R) = h$, $\varphi^\pi_j(I, H, R) = h_0$, and $\varphi^\pi_k(I, H, R) = h(i)$ (at Step 1 of the TTC algorithm, agent i points to house h, house h points to agent k, agent k points to house $h(i)$, and house $h(i)$ points to agent i, so house h is assigned to agent i and house $h(i)$ is assigned to agent k; and at Step 2, the null house h_0 is assigned to agent j).

We now consider a reduced problem (I', H', R') that is obtained from (I, H, R) by removing agent i with his allotment $\varphi^\pi_i(I, H, R) = h$. Therefore, $I' = \{j, k\}$, $H' = \{h(i)\}$ (now $\{h(i)\} = H'_V$), and preferences are $R' = (R^{H'}_j, R^{H'}_k)$. The TTC rule based on the priority structure π for problem (I', H', R') results in allotments $\varphi^\pi_j(I', H', R') = h(i)$ and $\varphi^\pi_k(I', H', R') = h_0$ (at Step 1 of the TTC algorithm agent j points to house $h(i)$ and house $h(i)$ points to agent j who is the highest ranked agent in I' for house $h(i)$, so house $h(i)$ is assigned to agent j; and at Step 2, the null house h_0 is assigned to agent k). For agent k we have $\varphi^\pi_k(I, H, R) = h(i) \neq h_0 = \varphi^\pi_k(I', H', R')$ and for agent j we have $\varphi^\pi_j(I, H, R) = h_0 \neq h(i) = \varphi^\pi_j(I', H', R')$. Hence, the TTC rule based on cyclic priority structure π violates *consistency*. $\square$

The fact that the TTC rule based on a cyclic priority structure in Example 3 is not *consistent* is not a coincidence: any TTC rule based on ownership-adapted priorities is *consistent* if and only if the priority structure is acyclic.

Theorem 1 (**φ^π: Consistency $\Leftrightarrow$ π is Acyclic**). *Let π be a priority structure that is adapted to the ownership structure and $\mathcal{D}$ be a closed domain. Then, the TTC rule based on π on $\mathcal{D}$, φ^π, is consistent if and only if π is acyclic.*

Proof. Let π be a priority structure that is adapted to the ownership structure and $\mathcal{D}$ be a closed domain.

Only If Part: Assume that φ^π is *consistent*. We show that then π is acyclic.

Assume for the sake of contradiction that π is cyclic. Hence, there exist agents $i, j, k \in \mathcal{I}$ and houses $h, h' \in \mathcal{H}$ with $h' \notin \{h(i), h(j), h(k)\}$ such that

$$\pi^h(i) < \pi^h(j) < \pi^h(k)$$

and

$$\pi^{h'}(k) < \pi^{h'}(i) \text{ and } \pi^{h'}(k) < \pi^{h'}(j).$$

Since π is adapted to the ownership structure, if $h \in \{h(i), h(j), h(k)\}$, then $h = h(i)$ (since among the three agents agent i has the top priority for house h).

Consider the problem (I, H, R) where $I = \{i, j, k\}$, $H \cup \{h_0\} = \{h, h', h(i), h(j), h(k), h_0\}$, and preferences over *individually-rational* houses are such that

- $R_i : h' \; P_i \; h \; R_i \; h(i) \; P_i \ldots,$
- $R_j : h \; P_j \; h' \; P_j \; h(j) \; P_j \ldots,$ and
- $R_k : h \; P_k \; h(k) \; P_k \ldots.$

Then, the TTC rule φ^π assigns h' to i, h to k (because agent i has the top priority for house h, agent k has the top priority for house h', and then they trade), and $h(j)$ to j. Next, consider the reduced problem $(I', H', R^{H'}_{I'})$ that is created when agent i leaves with his allotment h', i.e., $I' = \{j, k\}$ and $H' \cup \{h_0\} = \{h, h(i), h(j), h(k), h_0\}$. Now, the TTC rule φ^π assigns h to j (because agent j has the top priority for house h in I' and house h is the best house among H' for agent j) and $h(k)$ to k; contradicting *consistency*.

If Part: Assume that π is acyclic. We show that then φ^π is *consistent*.

Consider a problem $(I, H, R) \in \mathcal{D}$ and remove a set of agents $\widehat{I} \subsetneq I$ together with their allotment $\widehat{H}$ as well as some unassigned houses $\widetilde{H}$ to obtain a reduced problem $(I', H', R^{H'}_{I'}) = \left(I \setminus \widehat{I}, H \setminus (\widehat{H} \cup \widetilde{H}), R^{-(\widehat{H} \cup \widetilde{H})}_{-\widehat{I}}\right)$. That is, the occupied houses of existing tenants in I' is the set of occupied houses in H': $H'_O = \cup_{i \in I'_E} h(i)$. We will show that for each $j \in I'$, $\varphi^\pi_j(I, H, R) = \varphi^\pi_j(I', H', R^{H'}_{I'})$. It suffices to consider the following three cases (all other cases can be obtained by iteratively applying these three cases):

Case 1. Only one unassigned house is removed: for $h \in H_V$ and $h \notin \varphi^\pi_I(I, H, R)$, $\widehat{I} = \emptyset$, $\widehat{H} = \emptyset$, and $\widetilde{H} = \{h\}$.

Case 2. A "trading cycle" $[i_1, h(i_2), i_2, \ldots, i_K, h(i_1)]$ is removed: for $1 \leq k \leq K$ with $\varphi^\pi_{i_k}(I, H, R) = h(i_{k+1})$, taking indices modulo K, $\widehat{I} = \{i_1, \ldots, i_K\} \subseteq I_E$, $\widehat{H} = \{h(i_1), \ldots h(i_K)\} \subseteq H_O$, and $\widetilde{H} = \emptyset$.

Case 3. A recipient of a vacant house is removed together with his allotment: for $h \in H_V$ with $\varphi^\pi_i(I, H, R) = h$, $\widehat{I} = \{i\}$, $\widehat{H} = \{h\}$, and $\widetilde{H} = \emptyset$.

In Case 1, note that during any step of the TTC algorithm with priorities π applied to the problem (I, H, R), no agent points to house h; otherwise h would be assigned. Thus, removing h from the problem does not change the outcome of the algorithm and $\varphi^\pi(I, H, R) = \varphi^\pi(I', H', R^{H'}_{I'})$.

In Case 2, consider a preference profile $\widetilde{R} \in \mathcal{R}(I, H)$ such that $\widetilde{R}_{-\widehat{I}} = R_{-\widehat{I}}$ and for each $i \in \widehat{I}$ and each $h \in H' \setminus \{\varphi^\pi_i(I, H, R), h(i)\}$ we have

- $\widetilde{R}_i : \varphi^\pi_i(I, H, R) \; \widetilde{P}_i \; h(i) \; \widetilde{P}_i \; h.$

Starting from problem (I, H, R), if any of the agents $i \in \widehat{I}$ changes his preferences from R_i to $\widetilde{R}_i$, by *strategy-proofness* of the TTC rule, he will receive the same allotment before and after. Then, by *non-bossiness* of the TTC rule, the allotments of all other agents will also not change. This argument can be applied step by step for all agents in $\widehat{I}$ to move from problem (I, H, R) to problem $(I, H, \widetilde{R})$. Hence, by *group strategy-proofness* of the TTC rule, we have $\varphi^\pi(I, H, \widetilde{R}) = \varphi^\pi(I, H, R)$. By the definition of preferences $\widetilde{R}$, in the first step of the TTC algorithm with priorities π applied to the problem $(I, H, \widetilde{R})$, the trading cycle $[i_1, h(i_2), i_2, \ldots, i_K, h(i_1)]$ forms. After allocating houses according to this trading cycle and after removing it, the problem becomes the reduced problem $(I', H', R^{H'}_{I'})$. Note that other trading cycles that formed in Step 1 for problems $(I, H, \widetilde{R})$ will form again in Step 1 for the reduced problem $(I', H', R^{H'}_{I'})$. Thus, for each $i \in I'$ we have $\varphi^\pi_i(I, H, \widetilde{R}) = \varphi^\pi_i(I', H', R^{H'}_{I'})$. Since $\varphi^\pi(I, H, R) = \varphi^\pi(I, H, \widetilde{R})$ this concludes the proof for Case 2.

In Case 3, consider a preference profile $\widetilde{R} \in \mathcal{R}(I, H)$ such that $\widetilde{R}_i = R_i$ and for each $j \in I' = I \setminus \{i\}$ preferences $\widetilde{R}_j$ are obtained from R_j by making house h unacceptable, i.e., $h(j) \; \widetilde{P}_j \; h$ while leaving preferences over $H' = H \setminus \{h\}$ unchanged and thus $\widetilde{R}^{H'}_{I'} = R^{H'}_{I'}$.

Starting from problem (I, H, R), if any of the agents $j \in I'$ changes his preferences from R_j to $\widetilde{R}_j$, by *strategy-proofness* of the TTC rule, he will receive the same allotment before and after. Then, by *non-bossiness* of the TTC rule, the allotments of all other agents will also not change. This argument can be applied step by step for all agents in I' to move from problem (I, H, R) to problem $(I, H, \widetilde{R})$. Hence, by *group strategy-proofness* of the TTC rule, we have $\varphi^\pi(I, H, \widetilde{R}) = \varphi^\pi(I, H, R)$. During the TTC algorithm with priorities π applied to the problem $(I, H, \widetilde{R})$, a trading cycle including i and h forms. Let $[i_1, h_1, i_2, h_2, \ldots, i_K, h_K]$ with $i_1 = i, h_1 = h$ be this trading cycle and t the step in which it forms.

By the definition of preferences $\widetilde{R}$, the same trading cycles form in the first $t-1$ steps of the TTC algorithm with priorities π applied to the two problems $(I, H, \widetilde{R})$ and $(I', H', \widetilde{R}^{H'}_{I'})$. Next, we consider Step t of the TTC algorithm with priorities π in the two problems.

If $K = 1$, i.e., if agent i points to h and house h points to i in Step t in problem $(I, H, \widetilde{R})$, then the only difference between the two problems is that in problem $(I, H, \widetilde{R})$ we have an additional trading cycle consisting of i and h. Otherwise, the same trading cycles form in the two problems. Moreover, in the consecutive Steps $t+1, t+2, \ldots$ the same trading cycles form in the two problems. Thus, for each $j \in I' = I \setminus \{i\}$, we have $\varphi^{\pi}_j(I', H', R^{H'}_{I'}) = \varphi^{\pi}_j(I, H, \widetilde{R}) = \varphi^{\pi}_j(I, H, R)$.

If $K > 1$, then we show that in Step t of the TTC algorithm applied to $(I', H', R^{H'}_{I'})$ a trading cycle $i_2, h_2, \ldots, i_K, h_K$ forms. If this is true, then it follows immediately that all other trading cycles in the two problems are the same and moreover, in the consecutive Steps $t+1, t+2, \ldots$ the same trading cycles form in the two problems. To show that trading cycle $i_2, h_2, \ldots, i_K, h_K$ forms, it suffices to show that h_K points to i_2 in Step t of the TTC algorithm applied to problem $(I', H', R^{H'}_{I'})$. Suppose not and house h_K points to an agent $j \neq i_2$. Then, $\pi^{h_K}(i_1) < \pi^{h_K}(j) < \pi^{h_K}(i_2)$. Note however that $\pi^{h_1}(i_2) < \pi^{h_1}(j)$ and $\pi^{h_1}(i_2) < \pi^{h_1}(i_1)$, since otherwise $h_1 = h$ would not point to i_2 in Step t of the TTC algorithm with priorities π applied to problem $(I, H, \widetilde{R})$. Thus, there is a cycle in π, contradicting its acyclicity. $\square$

4 A Characterization

Our second main result is that TTC rules based on ownership-adapted acyclic priorities are the only rules that satisfy *Pareto-optimality*, *individual-rationality*, *strategy-proofness*, *reallocation-proofness*, and *consistency*.

Theorem 2 (**A Characterization of φ^{π}**)**.** *Let $\mathcal{D}$ be a closed domain. Then, a rule ϕ on $\mathcal{D}$ satisfies Pareto-optimality, individual-rationality, strategy-proofness, reallocation-proofness, and consistency if and only if there exists an ownership-adapted acyclic priority structure π such that $\phi = \varphi^{\pi}$.*

We prove Theorem 2 through a sequence of lemmata (which we prove in Appendix A). Let $\mathcal{D}$ be a closed domain such that for each $R \in \mathcal{R}(\mathcal{I}, \mathcal{H})$, $(\mathcal{I}, \mathcal{H}, R) \in \mathcal{D}$. Throughout the remainder of the section we assume that rule ϕ on $\mathcal{D}$ satisfies *Pareto-optimality*, *individual-rationality*, *strategy-proofness*, *reallocation-proofness*,[17] and *consistency*.

Using *Pareto-optimality* and *individual-rationality*, we derive a **priority structure** $\pi = (\pi^h)_{h \in \mathcal{H}}$ from ϕ.

For each house $h \in \mathcal{H}$ we call a preference profile in which each agent likes house h best and only finds houses in $\{h, h(i)\}$ *individually-rational* a **version of a maximal conflict preference profile** for h. Formally, $R^h \in \mathcal{R}(\mathcal{I}, \mathcal{H})$ is a version of a maximal conflict preference profile for h if for each $i \in \mathcal{I}$ and $h' \in \mathcal{H} \setminus \{h, h(i)\}$ we have

- $R^h_i : h \; R^h_i \; h(i) \; P^h_i \; h'$.

Note that there can be multiple versions of a maximal conflict preference profile for h that differ in the ranking of houses that are not *individually-rational*. After we have defined π^h we will show that the definition is independent of which versions of maximal conflict preference profiles we choose.

[17]However, note that *reallocation-proofness* is used only in the proof of Lemma 4.

We consider the problem $(\mathcal{I}, \mathcal{H}, R^h)$ where R^h is some version of a maximal conflict preference profile. By *Pareto-optimality*, for some $i \in \mathcal{I}$ we have $\phi_i(\mathcal{I}, \mathcal{H}, R^h) = h$. We assign the top priority of house h to agent i, i.e., $\pi^h(1) = i$.

Note that, by *individual-rationality*, if house h is occupied by an existing tenant i, then agent i will have the top priority at house h, i.e., if for some $i \in \mathcal{I}_E$, $h = h(i)$, then $\pi^{h(i)}(1) = i$. Hence, the priority structure π we are constructing will be adapted to the ownership structure, i.e., for each $i \in \mathcal{I}_E$, $\pi^{h(i)}(1) = i$.

We next remove agent $\pi^h(1) = i$ and consider the remaining maximal conflict problem $(\mathcal{I} \setminus \{i\}, \mathcal{H}, R^h_{-i})$. Again, by *Pareto-optimality*, for some $j \in \mathcal{I} \setminus \{i\}$ we have $\varphi_j(\mathcal{I} \setminus \{i\}, \mathcal{H}, R^h_{-i}) = h$. We assign the second priority of house h to agent j, i.e., $\pi^h(2) = j$. We next remove agent $\pi^h(2) = j$ and consider the remaining maximal conflict problem $(\mathcal{I} \setminus \{i, j\}, \mathcal{H}, R^h_{-\{i,j\}})$, etc. We iterate in this way until we have considered all agents in $\mathcal{I}$. In this way we obtain a priority ordering π^h for each $h \in \mathcal{H}$.

Next, we establish a sequence of lemmata about properties of π.

Lemma 1 (**Maximal Conflict Preference Profile Independence**). *Each version of a maximal conflict preference profile for a house together with rule ϕ induces the same priority ranking for the house.*

Lemma 2 (**Consistent Reduction of Maximal Conflict Preference Profiles**). *Let $i, j \in \mathcal{I}$ be two different agents and house $h \in \mathcal{H}$. Consider a problem $(I, H, R) \in \mathcal{D}$ such that $I = \{i, j\}$, $\{h, h(i), h(j)\} \subseteq H \cup \{h_0\}$, and $R \in \mathcal{R}(I, H)$ is a version of the maximal conflict preference profile of h restricted to I and H. Then, $\pi^h(i) < \pi^h(j)$ implies $\phi_i(I, H, R) = h$ and $\phi_j(I, H, R) = h(j)$.*

Lemma 3 (**Acyclicity for Vacant Houses**). *Let $i, j, k \in \mathcal{I}$ be three different agents and houses $h, h' \in \mathcal{H}$ that are not owned by any of the three agents, i.e., $h, h' \notin \{h(i), h(j), h(k)\}$. Then,*

$$\pi^h(i) < \pi^h(j) < \pi^h(k) \text{ implies } [\pi^{h'}(i) < \pi^{h'}(k) \text{ or } \pi^{h'}(j) < \pi^{h'}(k)].$$

Lemma 4 (**Acyclicity for Occupied Houses**). *Let $i, j, k \in \mathcal{I}$ be three different agents, house $h(i) \in \mathcal{H}$ is occupied by agent i, and house $h' \in \mathcal{H}$ is not owned by any of the three agents, i.e., $h' \notin \{h(i), h(j), h(k)\}$. Then,*

$$\pi^{h(i)}(i) < \pi^{h(i)}(j) < \pi^{h(i)}(k) \text{ implies } [\pi^{h'}(i) < \pi^{h'}(k) \text{ or } \pi^{h'}(j) < \pi^{h'}(k)].$$

Lemmata 3 and 4 together imply that our constructed **priority structure $\boldsymbol{\pi}$ is acyclic**. We are now ready to start proving that if a rule ϕ defined on a closed domain $\mathcal{D}$ satisfies *Pareto-optimality*, *individual-rationality*, *strategy-proofness*, *reallocation-proofness*, and *consistency*, then it is a TTC rule based on ownership-adapted acyclic priorities, i.e., $\phi = \varphi^\pi$. To this end, we first show that ϕ adapts to the priority structure π for top priority agents.

Lemma 5 (**Top Priority Adaptation**). *For each problem $(I, H, R) \in \mathcal{D}$, if agent $i \in I$ has the top priority in I for a vacant house $h \in H_V$, i.e., for each $j \in I \setminus \{i\}$, $\pi^h(i) < \pi^h(j)$, then $\phi_i(I, H, R) \; R_i \; h$.*

We now prove that rule ϕ is the TTC rule φ^π.

Proposition 2 ($\boldsymbol{\phi = \varphi^\pi}$)**.** *If a rule ϕ defined on a closed domain $\mathcal{D}$ satisfies Pareto-optimality, individual-rationality, strategy-proofness, reallocation-proofness, and consistency, then it is the TTC rule that is based on the ownership-adapted acyclic priority structure π, i.e., $\phi = \varphi^\pi$.*

Proof. Assume for the sake of contradiction that there is a problem $(I, H, R) \in \mathcal{D}$ such that $\phi(I, H, R) \neq \varphi^\pi(I, H, R)$. Consider the first Step t in which a trading cycle of agents $1, \ldots, K$ and houses $\varphi^\pi_1(I, H, R), \ldots, \varphi^\pi_K(I, H, R)$ forms according to φ^π that contains agents that are differently matched than at $\phi(I, H, R)$. Consider the reduced problem (I', H', R') of (I, H, R) where all trading cycles of the first $t - 1$ steps are removed. By *consistency*, for some $k \in \{1, \ldots, K\}$, $\phi_k(I', H', R') = \phi_k(I, H, R) \neq \varphi^\pi_k(I, H, R) = \varphi^\pi_k(I', H', R')$. Each agent $k \in \{1, \ldots, K\}$ prefers $\varphi^\pi_k(I', H', R')$ most among houses in H'.

For every $k \in \{1, \ldots, K\}$ we define preferences $\widetilde{R}_k \in \mathcal{R}(k, H')$ over *individually-rational* houses such that

- $\widetilde{R}_k : \varphi^\pi_k(I', H', R')\ \widetilde{P}_k\ \varphi^\pi_{k-1}(I', H', R')\ \widetilde{R}_k\ h(k)\ \widetilde{R}_k \ldots$ (modulo K),

e.g., by moving $\varphi^\pi_{k-1}(I', H', R')$ just after $\varphi^\pi_k(I', H', R')$ and, if $\varphi^\pi_{k-1}(I', H', R') \neq h(k)$ (i.e., when $\varphi^\pi_{k-1}(I', H', R')$ is vacant), by moving $h(k)$ just after $\varphi^\pi_{k-1}(I', H', R')$ without changing the ordering of other houses.

First, we consider the preference profile $R^0 = (\widetilde{R}_{\{1,\ldots,K\}}, R'_{-\{1,\ldots,K\}})$.

Let $k \in \{1, \ldots, K\}$. If $\varphi^\pi_{k-1}(I', H', R') = h(k)$, then by *individual-rationality*, we have $\phi_k(I', H', R^0)\ R_k\ \varphi^\pi_{k-1}(I', H', R')$ and $\phi_k(I', H', R^0) \in \{\varphi^\pi_{k-1}(I', H', R'), \varphi^\pi_k(I', H', R')\}$. If $\varphi^\pi_{k-1}(I', H', R') \neq h(k)$, then $\varphi^\pi_{k-1}(I', H', R')$ is a vacant house and agent k has the top priority for it. Hence, by Lemma 5, we have $\phi_k(I', H', R^0) R_k \varphi^\pi_{k-1}(I', H', R')$ and $\phi_k(I', H', R^0) \in \{\varphi^\pi_{k-1}(I', H', R'), \varphi^\pi_k(I', H', R')\}$. To summarize, for each $k \in \{1, \ldots, K\}$, $\phi_k(I', H', R^0) \in \{\varphi^\pi_{k-1}(I', H', R'), \varphi^\pi_k(I', H', R')\}$. So, $\phi_{\{1,\ldots,K\}}(I, H, R^0) = \varphi^\pi_{\{1,\ldots,K\}}(I', H', R')$. By *Pareto-optimality*, for each $k \in \{1, \ldots, K\}$, $\phi_k(I, H, R^0) = \varphi^\pi_k(I', H', R')$.

Next, let $\widehat{I} \subseteq \{1, \ldots, K\}$. By induction on the size of $\widehat{I}$ we will show that for each $k \in \{1, \ldots, K\}$, $\phi_k\big(I, H, (\widetilde{R}_{\{1,\ldots,K\}\setminus\widehat{I}}, R_{-(\{1,\ldots,K\}\setminus\widehat{I})})\big) = \varphi^\pi_k(I', H', R')$.

Induction Basis. If $|\widehat{I}| = 0$, i.e., $\widehat{I} = \emptyset$, then the claim holds as observed above.

Induction Hypothesis. We assume that for each $l \in \{1, \ldots, |K| - 1\}$, each $\widehat{I} \subsetneq \{1, \ldots, K\}$ with $|\widehat{I}| \leq l$, and each $R^l = (\widetilde{R}_{\{1,\ldots,K\}\setminus\widehat{I}}, R_{-(\{1,\ldots,K\}\setminus\widehat{I})})$, we have that for each $k \in \{1, \ldots, K\}$, $\phi_k(I, H, R^l) = \varphi^\pi_k(I', H', R')$.

Induction Step. Let $l \in \{0, \ldots, |K| - 1\}$, $\widehat{I} \subseteq \{1, \ldots, K\}$ with $|\widehat{I}| = l + 1$, and $R^{l+1} = (\widetilde{R}_{\{1,\ldots,K\}\setminus\widehat{I}}, R_{-(\{1,\ldots,K\}\setminus\widehat{I})})$. Let $j \in \widehat{I}$ and $R^l = (\widetilde{R}_{\{1,\ldots,K\}\setminus(\widehat{I}\setminus\{j\})}, R_{-(\{1,\ldots,K\}\setminus(\widehat{I}\setminus\{j\}))})$.

By the induction hypothesis, for each $k \in \{1, \ldots, K\}$, $\phi_k(I, H, R^l) = \varphi^\pi_k(I', H', R')$. We start by showing that $\phi_j(I, H, R^{l+1}) = \varphi^\pi_j(I', H', R')$. That is, when agent j changes his preferences from $\widetilde{R}_j$ to R_j at the preference profile R^l, house $\varphi^\pi_j(I', H', R')$ is still assigned to him under rule ϕ at the changed preference profile R^{l+1}. By *strategy-proofness* we have $\phi_j(I,H,R^{l+1})\ R_j\ \phi_j(I,H,R^l)$. Since $\phi_j(I,H,R^l)=\varphi^\pi_j(I',H',R')$ is agent j's best house at preference profiles R^l and R^{l+1}, we have $\phi_j(I,H,R^{l+1})=\phi_j(I,H,R^l)=\varphi^\pi_j(I',H',R')$.

Since the choice of agent $j\in\widehat{I}$ was arbitrary, we obtain $\phi_j(I,H,R^{l+1})=\varphi^\pi_j(I',H',R')$ for each $j\in\widehat{I}$. By *individual-rationality*, for each $k\in\{1,\ldots,K\}\setminus\widehat{I}$, $\phi_k(I,H,R^{l+1})\in\{\varphi^\pi_{k-1}(I',H',R'),\varphi^\pi_k(I',H',R')\}$. So, $\phi_{\{1,\ldots,K\}}(I,H,R^{l+1})=\varphi^\pi_{\{1,\ldots,K\}}(I',H',R')$. By *Pareto-optimality*, for each $k\in\{1,\ldots,K\}$, $\phi_k(I,H,R^{l+1})=\varphi^\pi_k(I',H',R')$.

Thus, if $\widehat{I}=\{1,\ldots,K\}$, then $\big(I,H,(\widetilde{R}_{\{1,\ldots,K\}\setminus\widehat{I}},R_{-(\{1,\ldots,K\}\setminus\widehat{I})})\big)=(I',H',R')$ and it follows by induction that for each $k\in\{1,\ldots,K\}$, $\phi_k(I',H',R')=\varphi^\pi_k(I',H',R')$; a contradiction. □

Finally, recall that by Proposition 1, the TTC rule φ^π satisfies *Pareto-optimality*, *individual-rationality*, *strategy-proofness*, and *reallocation-proofness* and by Theorem 1 and the acyclicity of π (Lemmata 3 and 4), φ^π is *consistent*. The proof of Theorem 2 is now complete. We prove the independence of properties used in the characterization (Theorem 2) in Appendix B.

5 Further Results

5.1 Results implied by Theorems 1 and 2

For house allocation problems with only existing tenants (on $\mathfrak{D}^E$), we obtain the following characterization of the core / TTC rule as a corollary; see Ma (1994, Theorem 1), Sönmez (1999, Corollary 3), and Svensson (1999, Theorem 2).

Corollary 1 (Characterizing the Core on $\mathfrak{D}^E$). *A rule for house allocation problems with only existing tenants (on $\mathfrak{D}^E$) is Pareto optimal, individually rational (for tenants), and strategy-proof if and only if it is the core.*

***Proof*.** For house allocation problems with only existing tenants (on $\mathfrak{D}^E$), the core is unique and it is the outcome of the TTC algorithm (Roth and Postlewaite, 1977, Theorem 2'). Furthermore, any priority structure is acyclic. In fact, the only information that matters (in the absence of vacant houses) is who owns which house. Thus, Lemma 4 immediately holds for house allocation problems with only existing tenants. Thus, since *reallocation-proofness* is only used in the proof of Lemma 4 (on acyclicity for occupied houses), it is now not needed in the proof of Proposition 2. Note also that Lemma 5 (top priority adaptation) is only used for vacant houses in the proof of Proposition 2. We now show that *consistency* is also not needed in the proof of Proposition 2 and hence, Propositions 1 and 2 imply the result.

The minor adjustments to the proof of Proposition 2 needed to avoid the use of *consistency* are marked in *italics*.

Proposition 2 Proof Adjustments without Consistency: Assume for the sake of contradiction that there is a problem *with only existing tenants* $(I,H,R)\in\mathfrak{D}^E$ such that

$\phi(I,H,R) \neq \varphi^{\pi}(I,H,R)$. Consider the first Step t in which a trading cycle of agents $1,\dots,K$ and houses $\varphi^{\pi}_1(I,H,R),\dots,\varphi^{\pi}_K(I,H,R)$ forms according to φ^{π} that contains agents that are differently matched than at ϕ. *Without loss of generality, we assume that also for all other problems* $(I,H,\widehat{R})$ *the first such cycle occurs in Step* t *or later. Otherwise, we may choose a different problem* (I,H,R) *with* $\phi(I,H,R) \neq \varphi^{\pi}(I,H,R)$ *to start with. By our choice of problem* (I,H,R)*, we now know that at all other problems* $(I,H,\widehat{R})$*, agents are always matched by* ϕ *according to the TTC rule in all earlier steps of the TTC algorithm.*

For every $k \in \{1,\dots,K\}$ we define preferences $\widetilde{R}_k \in \mathcal{R}(k,H)$ over *individually-rational* houses such that

- $\widetilde{R}_k : \varphi^{\pi}_k(I,H,R)\ \widetilde{P}_k\ h(k)\ \widetilde{R}_k \dots$ (modulo K),

e.g., by moving $h(k) = \varphi^{\pi}_{k-1}(I,H,R)$ just after $\varphi^{\pi}_k(I,H,R)$.

First, we consider the preference profile $R^0 = (\widetilde{R}_{\{1,\dots,K\}}, R_{-\{1,\dots,K\}})$. *Under* R^0*, by our choice of problem* (I,H,R) *and the definition of* φ^{π}*, during the first* $t-1$ *steps of the TTC algorithm the same trading cycles as under* R *occur and the associated allotments made to agents are the same under rules* ϕ *and* φ^{π} *(and equal to the allotments made under* R*).*

Let $k \in \{1,\dots,K\}$. *Then,* $\varphi^{\pi}_{k-1}(I,H,R) = h(k)$ *and* by *individual-rationality*, we have $\phi_k(I,H,R^0)\ R_k\ \varphi^{\pi}_{k-1}(I,H,R)$ and $\phi_k(I,H,R^0) \in \{\varphi^{\pi}_{k-1}(I,H,R), \varphi^{\pi}_k(I,H,R)\}$. So, $\phi_{\{1,\dots,K\}}(I,H,R^0) = \varphi^{\pi}_{\{1,\dots,K\}}(I,H,R)$. By *Pareto-optimality*, for each $k \in \{1,\dots,K\}$, $\phi_k(I,H,R^0) = \varphi^{\pi}_k(I,H,R)$.

The remaining induction argument that completes the proof does not use *consistency* (or any result relying on *consistency*) and hence remains valid (*problem* (I,H,R) *now plays the role of problem* (I',H',R')). □

Kesten (2006) studied house allocation problems with quotas and showed that the TTC rule adapted to the priority structure is *consistent* if and only if the priority structure satisfies Kesten's acyclicity condition (Kesten, 2006, Theorem 2). We obtain this result as a corollary when the quota of each house is one (we discuss the related literature on acyclic priority structures for house allocation with quotas problems in Appendix C.3).

Corollary 2 (TTC, Consistency, and Acyclicity on $\mathfrak{D}^N$). *A TTC rule adapted to the priority structure for house allocation problems with only new applicants (on* $\mathfrak{D}^N$*) is consistent if and only if the priority structure is Kesten acyclic.*

***Proof*.** For house allocation problems with only new applicants (on $\mathfrak{D}^N$) Kesten's scarcity condition of acyclicity is automatically satisfied. Thus our acyclicity condition is equal to Kesten acyclicity (on $\mathfrak{D}^N$). Then, the result follows from Theorem 1. □

For house allocation problems with only new applicants (on $\mathfrak{D}^N$), we call a TTC rule based on acyclic priorities an **efficient priority rule**.[18] For house allocation problems with only new applicants (on $\mathfrak{D}^N$), Ehlers and Klaus (2006, Proposition 2 and Theorem 1) characterized the class of efficient priority rules. We obtain their result as follows.

[18] A rule on $\mathfrak{D}^N$ adapts to the priority structure if and only if it chooses stable matchings, or equivalently, no justified envy occurs; see also Balinski and Sönmez (1999, Theorem 2). Furthermore, a rule on $\mathfrak{D}^N$

Corollary 3 (**Characterizing Efficient Priority Rules on $\mathfrak{D}^N$**)**.** *A rule for house allocation problems with only new applicants (on $\mathfrak{D}^N$) is Pareto-optimal, strategy-proof, and reallocation-consistent if and only if it is an efficient priority rule.*

Proof. For house allocation problems with only new applicants (on $\mathfrak{D}^N$), *consistency* implies *reallocation-consistency*. Furthermore, acyclicity coincides with Kesten acyclicity and thus the TTC rule based on acyclic priorities is the agents-proposing deferred acceptance rule with acyclic priorities (Kesten, 2006, Theorem 1). Since acyclicity also coincides with Ergin acyclicity, it follows that the agents-proposing deferred acceptance rule with acyclic priorities satisfies *Pareto-optimality* (Ergin, 2002, Theorem 1) and therefore is an efficient priority rule according to Ehlers and Klaus (2006). Thus, the class of TTC rules based on acyclic priorities is the class of efficient priority rules. Since *reallocation-proofness* is only used in the proof of Lemma 4 (on acyclicity for occupied houses), it is not needed in the proof of Theorem 2 for the domain of house allocation problems with only new applicants $\mathfrak{D}^N$. Now, Theorem 2 for $\mathfrak{D}^N$ implies that a rule on $\mathfrak{D}^N$ satisfies *Pareto-optimality*, *strategy-proofness*, and *reallocation-consistency* if and only if it is an efficient priority rule (note that for Theorem 2 only *individual-rationality for tenants* was necessary). □

5.2 A Representation of TTC Rules based on Ownership-Adapted Acyclic Priorities as Two-Step Rules

Recall that for house allocation problems with only new applicants (on $\mathfrak{D}^N$), we call a TTC rule based on acyclic priorities an efficient priority rule. An efficient priority rule has the property that at each step of the TTC algorithm, each trading cycle contains at most two agents and two houses. That is, at each step of the TTC algorithm, either the top priorities of remaining houses are assigned to exactly one agent (a dictator) or the top priorities of remaining houses are divided between two agents. If there is an agent (a dictator) who has top priority for all remaining houses then he gets his best house among the remaining houses. If the top priorities of remaining houses are divided between two agents, then, at that step of the TTC algorithm, either (i) both agents get a house for which they have the top priority, or (ii) they swap two houses for which they have top priorities, or (iii) only one of them gets one of his top priority houses and the other gets nothing. In the last case, the agent who gets nothing becomes a dictator at the next step of the TTC algorithm because of acyclicity of the priority structure.

Example 4 (**Efficient Priority Rules**)**.** For house allocation problems with only new applicants (on $\mathfrak{D}^N$), the class of *efficient priority rules* is the subclass of TTC rules where for each house allocation problem with only new applicants there is either one agent with

is an *efficient priority rule* if the assignment of houses to agents are determined by the agents-proposing deferred acceptance rule and it adapts to an (Ergin) acyclic priority structure. Finally, since Ergin and Kesten acyclicity coincide on $\mathfrak{D}^N$ (see Lemma 6 in Appendix C.3), the agents-proposing deferred acceptance rule based on acyclic priorities is the TTC rule based on acyclic priorities. See Appendix C.3 for further details.

the highest priority for all houses or there are two agents who share the first and second priorities of each house, i.e., the acyclic priority structure π is such that

- for each $(I, H, R) \in \mathfrak{D}^N$ there is an agent $i \in I$ such that for each $h \in H$, $\pi_I^h(1) = i$, or there are two agents $i, j \in I$ such that for each $h \in H$, $\{\pi_I^h(1), \pi_I^h(2)\} = \{i, j\}$. $\square$

For more general house allocation problems with existing tenants, TTC rules based on ownership-adapted acyclic priorities have the property that at each step of the TTC algorithm at most two *new applicants* and two *vacant houses* are involved in a trading cycle. A very natural subclass of TTC rules based on ownership-adapted acyclic priorities is the class of rules where at each step of the TTC algorithm, at most one trading cycle involving a vacant house appears and this trading cycle contains at most one new applicant and at most one vacant house. This class was introduced under the name of **YRMH-IGYT (you request my house - I get your turn) rules** by Abdulkadiroğlu and Sönmez (1999). Sönmez and Ünver (2010) showed that a rule satisfies *Pareto-optimality*, *individual-rationality*, *strategy-proofness*, *weak neutrality*, and *consistency* if and only if it is a YRMH-IGYT rule.

Example 5 (YRMH-IGYT Rules). The class of *YRMH-IGYT rules* is the subclass of TTC rules based on ownership-adapted acyclic priorities where for each problem there is a single agent who has top priority at each vacant house, i.e., the priority structure π is such that

(i) for each problem $(I, H, R) \in \mathcal{D}$ and for each $i \in I_E$, $\pi_I^{h(i)}(1) = i$ and

(ii) for each problem $(I, H, R) \in \mathcal{D}$ there exists an agent $i \in I$ such that for each $h \in H_V$, $\pi_I^h(1) = i$.

Another way to describe the set of YRMH-IGYT rule is as follows. Let π be a serial dictatorship priority structure. Then, for any $h, h' \in \mathcal{H}$, $\pi^h = \pi^{h'}$, i.e., every house has the same priority ordering. Let $\widehat{\pi}$ denote the priority structure obtained from π by adapting it to the ownership structure. It is easy to see that, since $\widehat{\pi}$ is based on a serial dictatorship priority structure, it is acyclic. Then, a rule ϕ is a YRMH-IGYT rule if and only if there exists a serial dictatorship priority structure π such that $\phi = \varphi^{\widehat{\pi}}$. $\square$

Next we give an alternative description of TTC rules based on ownership-adapted acyclic priorities in terms of two specific rules, YRMH-IGYT and efficient priority rules, that are applied in two steps. That is, the class of TTC rules based on ownership-adapted acyclic priorities is equivalent to a class of **two-step rules**. Essentially, these rules can be described as follows: The agents are split into two groups, where the first group contains all existing tenants and some new applicants and the second group consists of only new applicants. In the first step, houses are allocated to the first group of agents according to - essentially - the YRMH-IGYT rule. In the second step, houses that have not been allocated in the first step are allocated to the second group of agents according to an efficient priority rule.

We used the term "essentially" in the previous paragraph, because the allocation in the first step is generated according to a rule that might slightly differ from a YRMH-IGYT rule because we allow for the possibility that two agents have top priority at different vacant houses in steps of the TTC where only these two agents are left in the problem. Formally, we define a TTC rule ϕ^π based on ownership-adapted acyclic priorities π to be an **almost YRMH-IGYT rule** if

(i) for each problem $(I, H, R) \in \mathcal{D}$ and for each $i \in I_E$, $\pi_I^{h(i)}(1) = i$ and

(ii) for each problem $(I, H, R) \in \mathcal{D}$ **with** $|\boldsymbol{I}| > \mathbf{2}$ there exists an agent $i \in I$ such that for each $h \in H_V$, $\pi_I^h(1) = i$.

That is, for each problem each existing tenant has the top priority for his own house and for each problem with more than two agents there exists an agent who has top priority for all vacant houses. This means that, since domain $\mathcal{D}$ is closed, at each step of the TTC algorithm an existing tenant has the top priority for his own house and at each step of the TTC algorithm with more than two remaining agents there exists an agent who has top priority for all remaining vacant houses. Note that then the difference between a YRMH-IGYT rule and an almost YRMH-IGYT rule is that for the latter rule, the underlying priorities are only almost serial dictatorship priorities because the two lowest ranked agents might share ownership of remaining vacant houses. For problems with a large number of agents, such a rule behaves essentially like a YRMH-IGYT rule.

Example 6 (An almost YRMH-IGYT rule). Consider agents $\mathcal{I} := \{a, b, c, d, e\}$ with $\mathcal{I}_E = \{a, b, c\}$, houses $\{h(a), h(b), h(c), h_1, h_2, h_3, h_4, h_5\}$ and the following priority structure:

n	$\pi^{h(a)}(n)$	$\pi^{h(b)}(n)$	$\pi^{h(c)}(n)$	$\pi^{h_1}(n)$	$\pi^{h_2}(n)$	$\pi^{h_3}(n)$	$\pi^{h_4}(n)$	$\pi^{h_5}(n)$
1	a	b	c	d	d	d	d	d
2	d	d	d	b	b	b	b	b
3	b	c	b	c	c	c	c	c
4	c	a	e	e	a	a	e	e
5	e	e	a	a	e	e	a	a

Table 4: An acyclic priority structure: the top part of the priorities of Table 1.

Note that the TTC algorithm based on Table 4 priorities assigns allotments equal to the YRMH-IGYT rule for any Step t with $|I_t| > 2$. However, for a last Step t with $I_t = \{a, e\}$, the TTC algorithm might assign different allotments because agents a and e have different priorities at different vacant houses (the YRMH-IGYT rule would assign priorities dictatorially at that step as well). □

Definition 9 (Two-Step Rules). A rule ϕ defined on domain $\mathcal{D}$ is a **two-step rule** if there are

- a partition of the set of agents $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ such that the first group contains all tenants $\mathcal{I}_E \subseteq \mathcal{I}_1$ and
- rules ϕ^1 and ϕ^2 such that

rule ϕ^1 is an almost YRMH-IGYT rule that is defined for all problems $(I, H, R) \in \mathcal{D}$ with $I \subseteq \mathcal{I}_1$ and

rule ϕ^2 is a efficient priority rule that is defined for all problems $(I, H, R) \in \mathcal{D}$ with $I \subseteq \mathcal{I}_2$,

and these rules are applied in two steps as follows:

Step 1. for each $(I, H, R) \in \mathcal{D}$ and each $i \in I_1 := \mathcal{I}_1 \cap I$ we have $\phi_i(I, H, R) = \phi^1_i(I_1, H, R_{I_1})$ and

Step 2. for each $i \in I_2 := \mathcal{I}_2 \cap I$ we have $\phi_i(I, H, R) = \phi^2_i(I_2, H \setminus \widehat{H}, R_{I_2})$ where $\widehat{H} := \phi_{I_1}(I, H, R)$.

Example 7 (A TTC Rule based on Ownership-Adapted Acyclic Priorities as a Two-Step Rule). We reconsider the TTC rule based on ownership-adapted acyclic priorities π defined by Table 1 that we discussed in Example 2. It can be reinterpreted as a two-step rule as follows: To determine the first group of agents $\mathcal{I}_1$, we consider the lowest priority that any tenant has for any house. For the priorities π given in Table 1, this lowest priority of 5 is given to tenant a (e.g., for house h_1). Then, we consider all agents that have as least as high a priority at all houses: $\mathcal{I}_1 := \{j \in \mathcal{I} \mid \pi^h(j) \leq 5 \text{ for all } h \in \mathcal{H}\} = \{a, b, c, d, e\}$. The remaining agents form the second group $\mathcal{I}_2 := \{f, g, i\}$.

The rule ϕ^1 is the almost YRMH-IGYT rule based on Table 4 priorities previously described in Example 6. Rule ϕ^2 is now defined through the (acyclic) priorities consisting of the last three rows of Table 1:

n	$\pi^{h(a)}(n)$	$\pi^{h(b)}(n)$	$\pi^{h(c)}(n)$	$\pi^{h_1}(n)$	$\pi^{h_2}(n)$	$\pi^{h_3}(n)$	$\pi^{h_4}(n)$	$\pi^{h_5}(n)$
1	f	g	f	f	f	f	g	f
2	g	f	g	g	g	g	f	g
3	i	i	i	i	i	i	i	i

Table 5: An acyclic priority structure: the bottom part of the priorities of Table 1.

Note that for any problem $(I, H, R) \in \mathcal{D}$ with $I \subseteq \mathcal{I}_2$ the TTC rule based on Table 5 priorities is an efficient priority rule. $\square$

We now show that the correspondence between TTC rules based on ownership-adapted acyclic priorities and two-step rules holds in general.

Proposition 3 (Characterizing TTC Rules based on Ownership-Adapted Acyclic Priorities as Two-Step Rules). *The class of TTC rules based on ownership-adapted acyclic priorities is the class of two-step rules.*

Proof. Let ϕ^π be a TTC rule based on ownership-adapted acyclic priorities π. Consider the lowest priority assigned to a tenant at π, i.e., $m := \max_{h \in \mathcal{H}, i \in \mathcal{I}_E} \pi^h(i)$. Let $i^* \in \mathcal{I}_E$ be a tenant and $h^* \in \mathcal{H}$ be a house such that $\pi^{h^*}(i^*) = m$. We define the set $\mathcal{I}_1$ to be the set of agents who have higher or equal priority for house h^* than agent i^*, i.e., $\mathcal{I}_1 := \{j \in \mathcal{I} \mid \pi^{h^*}(j) \leq m\}$. The set $\mathcal{I}_2$ is the set of agents who have lower priority for house h^* than agent i^*, i.e., $\mathcal{I}_2 := \{j \in \mathcal{I} \mid \pi^{h^*}(j) > m\} = \mathcal{I} \setminus \mathcal{I}_1$. By definition, the two sets are disjoint, we have $\mathcal{I}_E \subseteq \mathcal{I}_1$, and $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$.

First we show that agents in $\mathcal{I}_1$ have higher priority than agents in $\mathcal{I}_2$ at all houses, i.e., for each $h \in \mathcal{H}$ we have $\mathcal{I}_1 = \{j \in \mathcal{I} \mid \pi^h(j) \leq m\}$ and $\mathcal{I}_2 = \{j \in \mathcal{I} \mid \pi^h(j) > m\}$. Consider $j \in \mathcal{I}_1 \setminus \{i^*\}$ and $k \in \mathcal{I}_2$ and assume by contradiction that there is a house $h \in \mathcal{H}$ with $\pi^h(k) < \pi^h(j)$. Since $\pi^{h^*}(j) < \pi^{h^*}(i^*) < \pi^{h^*}(k)$ and $k \notin \mathcal{I}_E$, acyclicity and $\pi^h(k) < \pi^h(j)$ imply that $\pi^h(i^*) < \pi^h(k) < \pi^h(j)$. Moreover, since $\pi^h(i^*) < \pi^h(k) < \pi^h(j)$ and $\pi^{h^*}(j) < \pi^{h^*}(i^*)$, acyclicity implies that h^* is occupied with j as a tenant. Since i^*'s priority at h^* is the lowest priority of any tenant at any house, tenant j cannot have lower priority at h than i^* has at h^*, i.e., $\pi^h(j) \leq \pi^{h^*}(i^*)$. Thus, there exists an agent ℓ with $\pi^{h^*}(j) < \pi^{h^*}(\ell) < \pi^{h^*}(i^*) < \pi^{h^*}(k)$ and $\pi^h(i^*) < \pi^h(k) < \pi^h(j) < \pi^h(\ell)$. But then agents k, j, ℓ with houses h, h^* form a cycle; a contradiction.

Next let $k \in \mathcal{I}_2$ and suppose there is a house $h \in \mathcal{H}$ with $\pi^h(k) < \pi^h(i^*)$. We may assume that there is another agent $j \in \mathcal{I}_1$ with $\pi^{h^*}(j) < \pi^{h^*}(i^*)$ and $\pi^h(i^*) < \pi^h(j)$. Otherwise i^* would have lower priority at h than at h^*. But since $k \notin \mathcal{I}_E$, we immediately get a contradiction with acyclicity.

Second, we show that the rule ϕ^π restricted to problems with agents in $\mathcal{I}_1$ is an almost YRMH-IGYT rule and restricted to problems with agents in $\mathcal{I}_2$ is an efficient priority rule. This will imply that ϕ^π is a two-step rule. Thus, we have to show that

- for each $(I, H, R) \in \mathcal{D}$ with $I \subseteq \mathcal{I}_1$ and $|I| > 2$ there is an agent $i \in I$ such that for each $h \in H_V$, $\pi_I^h(i) = 1$,
- for each $(I, H, R) \in \mathcal{D}$ with $I \subseteq \mathcal{I}_2$ there is an agent $i \in I$ such that for $h \in H$, $\pi_I^h(1) = i$, or there are two agents $i, j \in I$ such that for each $h \in H$, $\{\pi_I^h(1), \pi_I^h(2)\} = \{i, j\}$.

Since $\mathcal{I}_2$ contains only new applicants and the priority structure π is acyclic, the second item follows immediately from the fact that for house allocation problems with only new applicants the class of TTC rules based on acyclic priorities is the class of efficient priority rules. To show the first item, consider the problem (I', H', R') with $I' = I \cup \{i^*\}$, $H' = H \cup \{h(i^*), h^*\}$ and R' arbitrary. Let $j := \pi_{I'}^{h(i^*)}(2)$. We claim that at any $h \in H_V$ we have $\pi_I^h(j) = 1$. The claim trivially holds if $h = h(i^*)$. If $h \neq h(i^*)$, then by acyclicity we have $\pi_{I'}^h(1) \in \{i^*, j\}$. If $\pi_{I'}^h(1) = j$, then also $\pi_I^h(1) = j$ and we are finished. Thus, it suffices to show that $\pi_{I'}^h(1) = i^*$ yields a contradiction. Since $|I| > 2$, there is an agent $k \neq i^*, j$ in the problem (I', H', R'). Since $I' \subseteq \mathcal{I}_1$ and i^* has lowest priority among agents in $\mathcal{I}_1$ at house h^*, we have $\pi^{h^*}(k) < \pi^{h^*}(i^*)$ and $\pi^{h^*}(j) < \pi^{h^*}(i^*)$. Thus, if $\pi_{I'}^h(i^*) = 1$, then we have a cycle involving houses h, h^* and agents i^*, j, k and hence, a contradiction.

Finally, let ϕ be a two-step rule induced by rules ϕ^1 and ϕ^2. Let π^1 be the priority structure associated with rule ϕ^1 and π^2 be the priority structure associated with rule ϕ^2. Define a priority structure π as the concatenation of the two priority structures, i.e., for each

$h \in \mathcal{H}$ we let

$$\pi^h(i) = \begin{cases} \pi^{1,h}(i), & \text{if } i \in \mathcal{I}_1 \\ \pi^{2,h}(i) + |\mathcal{I}_1|, & \text{if } i \in \mathcal{I}_2. \end{cases}$$

Since π^1 and π^2 are acyclic priority structures, priority structure π is acyclic as well. Moreover, $\phi = \phi^\pi$. □

Appendix

A Proofs of Section 4 Lemmata

***Proof of Lemma 1* (Maximal Conflict Preference Profile Independence)**. Let R^h and $\bar{R}^h$ be different versions of a maximal conflict preference profile for h and π^h and $\bar{\pi}^h$ be the corresponding priority rankings obtained. We show that for each $i \in \{1, 2, \ldots, |\mathcal{I}|\}$ we have $\pi^h(i) = \bar{\pi}^h(i)$. We proceed by induction on i.

Induction Basis. Let $i = 1$ and suppose $\pi^h(1) \neq \bar{\pi}^h(1)$, i.e., $\phi_{\pi^h(1)}(\mathcal{I}, \mathcal{H}, R^h) = h$ but $\phi_{\bar{\pi}^h(1)}(\mathcal{I}, \mathcal{H}, \bar{R}^h) = h$. By *individual-rationality*, $\phi_{\bar{\pi}^h(1)}(\mathcal{I}, \mathcal{H}, R^h) = h(\bar{\pi}^h(1))$ and $\phi_{\pi^h(1)}(\mathcal{I}, \mathcal{H}, \bar{R}^h) = h(\pi^h(1))$.

First, consider the reduced problem (I, H, R) of $(\mathcal{I}, \mathcal{H}, R^h)$ where $I = \{\pi^h(1), \bar{\pi}^h(1)\}$, $H \cup \{h_0\} = \{h, h(\pi^h(1)), h(\bar{\pi}^h(1)), h_0\}$ and $R = (R^h)^H_I$. Note that $(I, H, R) \in \mathcal{D}$. By *consistency*, $\phi_{\pi^h(1)}(I, H, R) = h$.

Second, consider the reduced problem $(I, H, \bar{R})$ of $(\mathcal{I}, \mathcal{H}, \bar{R}^h)$ where I and H are defined as before and $\bar{R} = (\bar{R}^h)^H_I$. Note that $(I, H, \bar{R}) \in \mathcal{D}$. By *consistency*, $\phi_{\bar{\pi}^h(1)}(I, H, \bar{R}) = h$.

Third, starting from (I, H, R), change agent $\pi^h(1)$'s preferences to $\bar{R}_{\pi^h(1)}$. By *strategy-proofness*, $\phi_{\pi^h(1)}(I, H, (\bar{R}_{\pi^h(1)}, R_{\bar{\pi}^h(1)})) = h$. By *individual-rationality*, $\phi_{\bar{\pi}^h(1)}(I, H, (\bar{R}_{\pi^h(1)}, R_{\bar{\pi}^h(1)})) = h(\bar{\pi}^h(1))$.

Fourth, starting from $(I, H, (\bar{R}_{\pi^h(1)}, R_{\bar{\pi}^h(1)}))$ change agent $\bar{\pi}^h(1)$'s preferences to $\bar{R}_{\bar{\pi}^h(1)}$. This change results in preference profile $\bar{R}$. By *strategy-proofness*, $\phi_{\bar{\pi}^h(1)}(I, H, \bar{R}) = h(\bar{\pi}^h(1))$. By *Pareto-optimality*, $\phi_{\pi^h(1)}(I, H, \bar{R}) = h$; contradicting $\phi_{\bar{\pi}^h(1)}(I, H, \bar{R}) = h$.

Induction Hypothesis. We assume that for each $i' \leq i < |\mathcal{I}|$ we have $\pi^h(i') = \bar{\pi}^h(i')$.

Induction Step. We show that $\pi^h(i+1) = \bar{\pi}^h(i+1)$. Suppose $\pi^h(i+1) \neq \bar{\pi}^h(i+1)$.

Consider the problem $(I, \mathcal{H}, R^h_I)$ where $I = \mathcal{I} \setminus \{\pi^h(1), \ldots, \pi^h(i)\}$ and the problem $(\bar{I}, \mathcal{H}, \bar{R}^h_{\bar{I}})$ where $\bar{I} = \mathcal{I} \setminus \{\bar{\pi}^h(1), \ldots, \bar{\pi}^h(i)\}$. Note that $(I, \mathcal{H}, R^h_I), (\bar{I}, \mathcal{H}, \bar{R}^h_{\bar{I}}) \in \mathcal{D}$. By the induction assumption $I = \bar{I}$. Hence, $\phi_{\pi^h(i+1)}(I, \mathcal{H}, R^h_I) = h$ but $\phi_{\bar{\pi}^h(i+1)}(I, \mathcal{H}, \bar{R}^h_I) = h$. By *individual-rationality*, we have $\phi_{\bar{\pi}^h(i+1)}(I, \mathcal{H}, R^h_I) = h(\bar{\pi}^h(i+1))$ and $\phi_{\pi^h(i+1)}(I, \mathcal{H}, \bar{R}^h_{\bar{I}}) = h(\pi^h(i+1))$.

First, consider the reduced problem (I', H', R) of $(I, \mathcal{H}, R^h_I)$ where $I' = \{\pi^h(i+1), \bar{\pi}^h(i+1)\}$, $H' \cup \{h_0\} = \{h, h(\pi^h(i+1)), h(\bar{\pi}^h(i+1)), h_0\}$ and $R = (R^h)^{H'}_{I'}$. By *consistency*, $\phi_{\pi^h(i+1)}(I', H', R') = h$.

Second, consider the reduced problem $(I', H', \bar{R})$ of $(\bar{I}, \mathcal{H}, \bar{R}^h_{\bar{I}})$ where I' and H' are defined as before and $\bar{R} = (\bar{R}^h)^{H'}_{I'}$. By *consistency*, $\phi_{\bar{\pi}^h(i+1)}(I', H', \bar{R}) = h$.

An analog argument as for the induction basis shows that when changing preferences step by step from R to $\bar{R}$, *Pareto-optimality*, *individual-rationality*, and *strategy-proofness* imply that $\phi_{\pi^h(i+1)}(I, H, \bar{R}) = h$; contradicting $\phi_{\bar{\pi}^h(i+1)}(I, H, \bar{R}) = h$. $\square$

Proof of Lemma 2 **(Consistent Reduction of Maximal Conflict Preference Profiles)**. Let $i, j \in \mathcal{I}$ be two different agents and house $h \in \mathcal{H}$ and problem $(I, H, R) \in \mathcal{D}$ be such that $I = \{i, j\}$, $\{h, h(i), h(j)\} \subseteq H \cup \{h_0\}$, and $R \in \mathcal{R}(I, H)$ is a version of the maximal conflict preference profile of h restricted to I and H. We show that $\pi^h(i) < \pi^h(j)$ implies $\phi_i(I, H, R) = h$ and $\phi_j(I, H, R) = h(j)$.

Recall that π^h is generated by a version of a maximal conflict preference profile for house h. By Lemma 1, it is no loss of generality to assume that this maximal conflict preference profile is a preference profile $R^h \in \mathcal{R}(\mathcal{I}, \mathcal{H})$ such that R is its restriction to I and H, i.e., $(R^h)^H_I = R$.

By our construction to calibrate π^h, there exists a set of agents $\widetilde{I} := \big(\mathcal{I} \setminus \{\pi^h(1), \ldots, \pi^h(l)\}\big)$ such that $i, j \in \widetilde{I}$, agent i has the highest priority for house h in $\widetilde{I}$, i.e., $\pi^h(l+1) = i$, and $\phi_i(\widetilde{I}, \mathcal{H}, R^h_{\widetilde{I}}) = h$. By *individual-rationality*, for all $k \in \widetilde{I} \setminus \{i\}$ we have $\phi_k(\widetilde{I}, \mathcal{H}, R^h_{\widetilde{I}}) = h(k)$; in particular, $\phi_j(\widetilde{I}, \mathcal{H}, R^h_{\widetilde{I}}) = h(j)$.

Note that (I, H, R) is a reduced problem of $(\widetilde{I}, \mathcal{H}, R^h_{\widetilde{I}})$ obtained by removing all agents $\widetilde{I} \setminus \{i, j\}$ with their allotments $\bigcup_{k \in \left(\widetilde{I} \setminus \{i,j\}\right)} h(k) = \widetilde{H}$ and also by removing all unassigned houses $\widetilde{h} \in (\mathcal{H} \setminus \widetilde{H}) \setminus H$ that are not occupied by remaining agents i and j, i.e., $\widetilde{h} \notin \{h(i), h(j)\}$. By *consistency*, $\phi_i(I, H, R) = \phi_i(\widetilde{I}, \mathcal{H}, R^h_{\widetilde{I}})$ and $\phi_j(I, H, R) = \phi_j(\widetilde{I}, \mathcal{H}, R^h_{\widetilde{I}})$. Hence, $\phi_i(I, H, R) = h$ and $\phi_j(I, H, R) = h(j)$. $\square$

Proof of Lemma 3 **(Acyclicity for Vacant Houses)**. Let $i, j, k \in \mathcal{I}$ be three different agents and houses $h, h' \in \mathcal{H}$ that are not owned by any of the three agents, i.e., $h, h' \notin \{h(i), h(j), h(k)\}$. We show that $\pi^h(i) < \pi^h(j) < \pi^h(k)$ implies $[\pi^{h'}(i) < \pi^{h'}(k)$ or $\pi^{h'}(j) < \pi^{h'}(k)]$.

Assume for the sake of contradiction that $\pi^h(i) < \pi^h(j) < \pi^h(k)$, $\pi^{h'}(k) < \pi^{h'}(i)$ and $\pi^{h'}(k) < \pi^{h'}(j)$. Consider the problem (I, H, R) where $I = \{i, j, k\}$, $H \cup \{h_0\} = \{h, h', h(i), h(j), h(k), h_0\}$, and preferences over *individually-rational* houses are such that

- $R_i : h' \; P_i \; h \; P_i \; h(i) \; P_i \ldots$,
- $R_j : h \; P_j \; h(j) \; P_j \ldots$, and
- $R_k : h \; P_k \; h' \; P_k \; h(k) \; P_k \ldots$.

By *Pareto-optimality* and *individual-rationality*, either $\phi_i(I, H, R) = h'$ or $\phi_k(I, H, R) = h'$.

Case 1. $\phi_i(I, H, R) = h'$.

By *Pareto-optimality*, either $\phi_j(I, H, R) = h$ or $\phi_k(I, H, R) = h$.

Case 1.1. If $\phi_j(I,H,R)=h$, then consider the reduced problem $(I',H',R_{I'}^{H'})$ that is created when agent j leaves with his allotment h and furthermore (by *individual-rationality*) the unassigned house $h(j)$ is deleted from the problem, i.e., $I'=\{i,k\}$ and $H'\cup\{h_0\}=\{h',h(i),h(k),h_0\}$. By *consistency*, $\phi_i(I',H',R_{I'}^{H'})=h'$. However, note that $R_{I'}^{H'}$ is the restriction of a maximal conflict preference profile for h' to I' and H'. Hence, by Lemma 2, $\pi^{h'}(k)<\pi^{h'}(i)$ implies $\phi_k(I',H',R_{I'}^{H'})=h'$; a contradiction.

Case 1.2. If $\phi_k(I,H,R)=h$, then consider the reduced problem $(I',H',R_{I'}^{H'})$ that is created when agent i leaves with his allotment h' and furthermore (by *individual-rationality*) the unassigned house $h(i)$ is deleted from the problem, i.e., $I'=\{j,k\}$ and $H'\cup\{h_0\}=\{h,h(j),h(k),h_0\}$. By *consistency*, $\phi_k(I',H',R_{I'}^{H'})=h$. However, note that $R_{I'}^{H'}$ is the restriction of a maximal conflict preference profile for h to I' and H'. Hence, by Lemma 2, $\pi^{h}(j)<\pi^{h}(k)$ implies $\phi_j(I',H',R_{I'}^{H'})=h$; a contradiction.

Case 2. $\phi_k(I,H,R)=h'$.

By *Pareto-optimality*, $\phi_j(I,H,R)=h$. Consider the reduced problem $(I',H',R_{I'}^{H'})$ that is created when agent k leaves with his allotment h' and furthermore (by *individual-rationality*) the unassigned house $h(k)$ is deleted from the problem, i.e., $I'=\{i,j\}$ and $H'\cup\{h_0\}=\{h,h(i),h(j),h_0\}$. By *consistency*, $\phi_j(I',H',R_{I'}^{H'})=h$. However, note that $R_{I'}^{H'}$ is the restriction of a maximal conflict preference profile for h to I' and H'. Hence, by Lemma 2, $\pi^{h}(i)<\pi^{h}(j)$ implies $\phi_i(I',H',R_{I'}^{H'})=h$; a contradiction. □

***Proof of Lemma 4* Acyclicity for Occupied Houses**. Let $i,j,k\in\mathcal{I}$ be three different agents, $h(i)\in\mathcal{H}$ is occupied by agent i, and house $h'\in\mathcal{H}$ is not owned by any of the three agents, i.e., $h'\notin\{h(i),h(j),h(k)\}$. We show that $\pi^{h(i)}(i)<\pi^{h(i)}(j)<\pi^{h(i)}(k)$ implies $[\pi^{h'}(i)<\pi^{h'}(k)$ or $\pi^{h'}(j)<\pi^{h'}(k)]$.

Assume for the sake of contradiction that $\pi^{h(i)}(i)<\pi^{h(i)}(j)<\pi^{h(i)}(k)$, $\pi^{h'}(k)<\pi^{h'}(i)$, and $\pi^{h'}(k)<\pi^{h'}(j)$. Consider the problem (I,H,R) where $I=\{i,j,k\}$, $H\cup\{h_0\}=\{h',h(i),h(j),h(k),h_0\}$, and preferences over *individually-rational* houses are such that

- $R_i: h'\;P_i\;h(i)\;P_i\ldots$,
- $R_j: h(i)\;P_j\;h(j)\;P_j\ldots$, and
- $R_k: h(i)\;P_k\;h'\;P_k\;h(k)\;P_k\ldots$.

By *Pareto-optimality* and *individual-rationality*, either $\phi_i(I,H,R)=h'$ or $\phi_k(I,H,R)=h'$.

Case 1. $\phi_i(I,H,R)=h'$.

By *Pareto-optimality*, either $\phi_j(I,H,R)=h(i)$ or $\phi_k(I,H,R)=h(i)$.

Case 1.1. If $\phi_j(I,H,R)=h(i)$, then $\phi_k(I,H,R)=h(k)$. Now, consider $\widetilde{R}_i\in\mathcal{R}(i,H)$, $\widetilde{R}_j\in\mathcal{R}(j,H)$, and $\widetilde{R}_k\in\mathcal{R}(k,H)$ such that

- $\widetilde{R}_i: h(i)\;\widetilde{P}_i\ldots$,
- $\widetilde{R}_j: h'\;\widetilde{P}_j\;h(j)\;P_k\ldots$, and

- $\widetilde{R}_k : h' \, \widetilde{P}_k \, h(k) \, P_k \ldots$.

First, consider $R^1 = (R_i, R_j, \widetilde{R}_k)$. By *strategy-proofness*, $\phi_k(I, H, R^1) = h(k)$ and by *consistency*, $\phi_i(I, H, R^1) = h'$ and $\phi_j(I, H, R^1) = h(i)$. Second, consider $R^2 = (\widetilde{R}_i, \widetilde{R}_j, \widetilde{R}_k)$. By *individual-rationality*, $\phi_i(I, H, R^2) = h(i)$. Then, *consistency*, Lemma 2, and $\pi^{h'}(k) < \pi^{h'}(j)$ imply $\phi_k(I, H, R^2) = h'$ and $\phi_j(I, H, R^2) = h(j)$. We now show that agents i and j by changing their preferences from $(\widetilde{R}_i, \widetilde{R}_j)$ at R^2 to (R_i, R_j) at R^1 cause a violation of *reallocation-proofness*.

Consider agent i changing his preferences at R^2 from $\widetilde{R}_i$ to R_i. The resulting preference profile $R^3 = (R_i, \widetilde{R}_j, \widetilde{R}_k)$ is a version of a maximal conflict preference profile for house h' and Lemmata 1 and 2 together with $\pi^{h'}(k) < \pi^{h'}(i)$ and $\pi^{h'}(k) < \pi^{h'}(j)$ imply $\phi_k(I, H, R^3) = h'$ and $\phi_i(I, H, R^3) = \phi_i(I, H, R^2) = h(i)$. Hence, agent i does not change his allotment by unilaterally moving from R^2 to R^3.

Consider agent j changing his preferences at R^2 from $\widetilde{R}_j$ to R_j. The resulting preference profile is $R^4 = (\widetilde{R}_i, R_j, \widetilde{R}_k)$. By *individual-rationality*, $\phi_i(I, H, R^4) = h(i)$. Hence, by *individual-rationality*, $\phi_j(I, H, R^4) = h(j)$. Hence, agent j does not change his allotment by unilaterally moving from R^2 to R^4.

Finally, consider both agents i and j changing their preferences at the same time, moving from R^2 to R^1, and then swapping their allotments. Then, agent i receives the same allotment $\phi_j(I, H, R^1) = h(i) = \phi_i(I, H, R^2)$ while agent j is better off receiving $\phi_i(I, H, R^1) = h' \, \widetilde{P}_j \, h(j) = \phi_j(I, H, R^2)$; a contradiction to *reallocation-proofness*.

Case 1.2. If $\phi_k(I, H, R) = h(i)$, then consider the reduced problem $(I', H', R^{H'}_{I'})$ that is created when agent i leaves with his allotment h', i.e., $I' = \{j, k\}$ and $H' \cup \{h_0\} = \{h(i), h(j), h(k), h_0\}$. By *consistency*, $\phi_k(I', H', R^{H'}_{I'}) = h$. However, note that $R^{H'}_{I'}$ is the restriction of a maximal conflict preference profile for h to I' and H'. Hence, by Lemma 2, $\pi^{h(i)}(j) < \pi^{h(i)}(k)$ implies $\phi_j(I', H', R^{H'}_{I'}) = h(i)$; a contradiction.

Case 2. $\phi_k(I, H, R) = h'$.

By *individual-rationality*, $\phi_i(I, H, R) = h(i)$, contradicting *Pareto-optimality* (agents i and k would like to swap allotments). □

***Proof of Lemma 5* (Top Priority Adaptation)**. Let $(I, H, R) \in \mathcal{D}$ and assume that agent $i \in I$ has the top priority in I for a vacant house $h \in H_V$, i.e., for each $j \in I \setminus \{i\}$, $\pi^h(i) < \pi^h(j)$. We show that then $\phi_i(I, H, R) \, R_i \, h$.

Assume for the sake of contradiction that $h \, P_i \, \phi_i(I, H, R)$. By *strategy-proofness*, assume without loss of generality that agent i's preferences are the reduction of a version of the maximal conflict preferences for h, i.e., $R_i = \left(R^h_i\right)^H$ (if that wasn't the case, then agent i could switch to maximal conflict preferences and still not receive house h). By *individual-rationality*, $h \neq h(i)$ and $\phi_i(I, H, R) = h(i)$.

By *Pareto-optimality*, there is an agent j_1 such that $\phi_{j_1}(I, H, R) = h$. By *strategy-proofness*, assume without loss of generality that agent j_1's preferences are the reduction of a version of the maximal conflict preferences for h, i.e., $R_{j_1} = \left(R^h_{j_1}\right)^H$ (if that wasn't the case, then agent j_1 could switch to maximal conflict preferences and still receive house h while agent i still does not receive it).

In addition, assume that $(I, H, R) \in \mathcal{D}$ is a smallest such problem in terms of the number of agents, i.e., there is no problem with fewer agents and with a priority violation involving a top priority agent. This assumption and *consistency* imply that there are agents $j_1, \ldots, j_l$ (possibly $l = 1$) such that $\phi_{j_1}(I, H, R) = h$, $\phi_{j_2}(I, H, R) = h(j_1) \in H_O$, $\phi_{j_3}(I, H, R) = h(j_2) \in H_O$, $\ldots$, $\phi_{j_l}(I, H, R) = h(j_{l-1}) \in H_O$ and either $h(j_l) = h_0$ or $h(j_l) \in H_O$; but there is no $k \in I$ such that $\phi_k(I, H, R) = h(j_l)$. Furthermore, $I = \{i, j_1, \ldots, j_l\}$ (if not, by *consistency*, we could reduce the problem and obtain a top priority violation with fewer agents).

Case 1. $l = 1$ and $I = \{i, j_1\}$.

Recall that agents i and j_1's preferences are the reduced maximal conflict preferences used in the construction of π, i.e., R is the restriction of the maximal conflict preference profile used in the construction of π to agents I and houses H, i.e., $R = \left(R^h\right)_I^H$. Hence, by Lemma 2, $\phi_i(I, H, R) = h$; a contradiction.

Case 2. $l > 1$ and $I = \{i, j_1, \ldots, j_l\}$.

Let R'_{j_2} equal the restricted maximal conflict preferences used in the construction of $\pi^{h(j_1)}$, i.e., $R'_{j_2} = \left(R^{h(j_1)}_{j_2}\right)^H$ and define $R^1 = (R'_{j_2}, R_{-j_2})$. By *strategy-proofness*, $\phi_{j_2}(I, H, R^1) = h(j_1)$. This implies that $\phi_{j_1}(I, H, R^1) = h$ (if not, then by *individual-rationality* agent j_1's allotment would be his second best house $h(j_1)$, which is already allocated to agent j_2). *Pareto-optimality* now implies $\phi(I, H, R^1) = \phi(I, H, R)$.

Let R'_{j_3} equal the restricted maximal conflict preferences used in the construction of $\pi^{h(j_2)}$, i.e., $R'_{j_3} = \left(R^{h(j_2)}_{j_3}\right)^H$ and define $R^2 = (R'_{j_3}, R^1_{-j_3})$. By *strategy-proofness*, $\phi_{j_3}(I, H, R^1) = h(j_2)$. This implies that $\phi_{j_2}(I, H, R^1) = h(j_1)$ (if not, then by *individual-rationality* agent j_2's allotment would be his second best house $h(j_2)$, which is already allocated to agent j_3) and $\phi_{j_1}(I, H, R^1) = h$ (if not, then by *individual-rationality* agent j_1's allotment would be his second best house $h(j_1)$, which is already allocated to agent j_2). *Pareto-optimality* now implies $\phi(I, H, R^2) = \phi(I, H, R^1) = \phi(I, H, R)$.

We continue to replace the preferences of agents $j_4, \ldots, j_l$ with restricted maximal conflict preferences until we reach a restricted maximal conflict preference profile $R' := R^{|I|-2}$ with $R'_i = \left(R^h_i\right)^H$, $R'_{j_1} = \left(R^h_{j_1}\right)^H$, $R'_{j_2} = \left(R^{h(j_1)}_{j_2}\right)^H$, $\ldots$, $R'_{j_l} = \left(R^{h(j_{l-1})}_{j_l}\right)^H$, and $\phi(I, H, R') = \phi(I, H, R)$.

Next, consider $\widehat{R}_i$ that is obtained from R'_i by moving $h(j_1)$ between the best house h and the second best house $h(i)$, i.e., $\widehat{R}_i : h\, \widehat{P}_i\, h(j_1)\, \widehat{P}_i\, h(i)\, \widehat{R}_i\, h_0\, \widehat{P}_i \ldots$. By *strategy-proofness*, $\phi_i(I, H, (\widehat{R}_i, R'_{-i})) \neq h$ and by *individual-rationality*, $\phi_i(I, H, (\widehat{R}_i, R'_{-i})) \in \{h(j_1), h(i)\}$.

If $\phi_i(I, H, (\widehat{R}_i, R'_{-i})) = h(j_1)$, then by *individual-rationality* and *Pareto-optimality*, $\phi_{j_1}(I, H, (\widehat{R}_i, R'_{-i})) = h$, $\phi_{j_2}(I, H, (\widehat{R}_i, R'_{-i})) = h(j_2)$, $\phi_{j_3}(I, H, (\widehat{R}_i, R'_{-i})) = h(j_3)$, $\ldots$, $\phi_{j_l}(I, H, (\widehat{R}_i, R'_{-i})) = h(j_l)$. Now, consider the reduced problem where agents $j_2, \ldots, j_l$

leave with their allotments that are the houses they occupy, i.e., $\widehat{I} = \{i, j_1\}$ and $\widehat{H} = H \setminus \{h(j_2), \ldots, h(j_l)\}$. Since domain $\mathcal{D}$ is closed, $(\widehat{I}, \widehat{H}, \left(\widehat{R}_i, R'_{j_1}\right)^{\widehat{H}}) \in \mathcal{D}$. However, this is a problem with fewer agents than problem (I, H, R) with a priority violation involving a top priority agent; a contradiction.

Hence, $\phi_i(I, H, (\widehat{R}_i, R'_{-i})) = h(i)$. Since agent j_1 owns house $h(j_1)$, $\pi^{h(j_1)}(j_1) < \pi^{h(j_1)}(i)$ and $\pi^{h(j_1)}(j_1) < \pi^{h(j_1)}(j_2)$. If $\pi^{h(j_1)}(j_1) < \pi^{h(j_1)}(j_2) < \pi^{h(j_1)}(i)$, then by acyclicity (Lemmata 3 and 4) $\pi^h(j_1) < \pi^h(i)$ or $\pi^h(j_2) < \pi^h(i)$; contradicting that agent i has the top priority for house h, i.e., $\pi^h(i) < \pi^h(j_1)$ and $\pi^h(i) < \pi^h(j_2)$. Hence, $\pi^{h(j_1)}(j_1) < \pi^{h(j_1)}(i) < \pi^{h(j_1)}(j_2)$.

Now, consider the reduced problem where agents j_1 leaves with his allotment house h, i.e., $\overline{I} = I \setminus \{j_1\}$ and $\overline{H} = H \setminus \{h\}$. Since domain $\mathcal{D}$ is closed, $(\overline{I}, \overline{H}, \left(\widehat{R}_i, R'_{-i}\right)_{\overline{I}}^{\overline{H}}) \in \mathcal{D}$. By *consistency*, $\phi_{j_2}(\overline{I}, \overline{H}, \left(\widehat{R}_i, R'_{-i}\right)_{\overline{I}}^{\overline{H}}) = h(j_1)$. However, this is now a problem with fewer agents than problem (I, H, R) with priority violation involving a top priority agent; a contradiction. □

B Independence of Properties in Theorem 2

For each of the examples introduced to establish independence below we indicate the property of Theorem 2 it fails (while it satisfies all remaining properties).

Pareto-Optimality. The null rule ϕ^0 on $\mathcal{D}$ assigns to each tenant his occupied house and to each new applicant the null house. Hence, for each problem $(I, H, R) \in \mathcal{D}$ and each agent $i \in I$, $\phi^0_i(I, H, R) = h(i)$. The null rule ϕ^0 satisfies *individual-rationality*, *strategy-proofness*, *reallocation-proofness*, and *consistency*, but it violates *Pareto-optimality*. □

Individual Rationality (for Tenants). Let π be a priority structure such that for any $h, h' \in \mathcal{H}$, $\pi^h = \pi^{h'}$, i.e., every house has the same priority ordering or serial dictatorship ordering. The serial dictatorship rule φ^π on $\mathcal{D}$ now works as follows. For each problem $(I, H, R) \in \mathcal{D}$, the highest serial dictatorship priority agent in I, let's say agent i, is assigned his best house in $H \cup \{h_0\}$, the highest serial dictatorship priority agent in $I \setminus \{i\}$, let's say agent j, is assigned his best house in $(H \setminus \{\varphi^\pi_i(I, H, R)\}) \cup \{h_0\}$, and so on. The serial dictatorship rule φ^π satisfies *Pareto-optimality*, *strategy-proofness*, *reallocation-proofness*, and *consistency*, but it violates *individual-rationality for tenants* since the serial dictatorship priority structure is not adapted to the ownership structure. □

Strategy-Proofness. Let π be a priority structure such that for any $h, h' \in \mathcal{H}$, $\pi^h = \pi^{h'}$, i.e., every house has the same priority ordering or serial dictatorship ordering. Furthermore assume that all tenants have higher priority than all new applicants, i.e., for each house $h \in \mathcal{H}$, each tenant $i \in \mathcal{I}_E$, and each new applicant $j \in \mathcal{I}_N$, $\pi^h(i) < \pi^h(j)$. Let $\widehat{\pi}$ denote the priority structure obtained from π by adapting it to the ownership structure. Note that at $\widehat{\pi}$, again, all tenants have higher priority for all houses than all new applicants and $\widehat{\pi}$ is acyclic.

We define rule $\widehat{\phi}$ as follows. For each problem $(I, H, R) \in \mathcal{D}$, we first consider the problem consisting of existing tenants and all houses, i.e., we consider the problem $(I_E, H, R_{I_E}) \in \mathcal{D}$, and we apply the TTC rule $\varphi^{\widehat{\pi}}$, i.e.,

$$\text{for each } i \in I_E,\ \widehat{\phi}_i(I, H, R) = \varphi_i^{\widehat{\pi}}(I_E, H, R_{I_E}).$$

Since the TTC rule $\varphi^{\widehat{\pi}}$ is used, we have *Pareto-optimality*, *individual-rationality*, *strategy-proofness*, *reallocation-proofness*, and *consistency* among tenants.

Next, we apply the immediate acceptance algorithm to determine the matching for the remaining reduced problem $(I \setminus I_E, H \setminus \widehat{H}, R_{I \setminus I_E}^{H \setminus \widehat{H}})$ where $\widehat{H} = \widehat{\phi}_{I_E}(I, H, R)$.

Immediate Acceptance Algorithm

Step 1: Each new applicant applies to his favorite house in $\left(H \setminus \widehat{H}\right) \cup \{h_0\}$. Each house in $H \setminus \widehat{H}$ accepts the highest priority applicant and rejects all others. The null house h_0 accepts all applicants.

Step $r \geq 2$: Each new applicant who was rejected at Step $r-1$ applies to his favorite house in $\left(H \setminus \widehat{H}\right) \cup \{h_0\}$ that did not reject him yet. Each house in $H \setminus \widehat{H}$ not assigned in a previous step accepts the highest priority applicant and rejects all others. Each house in $H \setminus \widehat{H}$ that was assigned in a previous step rejects all applicants and the null house h_0 accepts all applicants.

The algorithm terminates when each new applicant in $I \setminus I_E$ is accepted by a house in $\left(H \setminus \widehat{H}\right) \cup \{h_0\}$. The matching where each agent is assigned the house that he was accepted by at the end of the algorithm is called the *immediate acceptance matching* and denoted by $IA^{\widehat{\pi}}(I \setminus I_E, H \setminus \widehat{H}, R_{I \setminus I_E}^{H \setminus \widehat{H}})$. Hence,

$$\text{for each } i \in I_N,\ \widehat{\phi}_i(I, H, R) = IA^{\widehat{\pi}}(I \setminus I_E, H \setminus \widehat{H}, R_{I \setminus I_E}^{H \setminus \widehat{H}}).$$

Any immediate acceptance rule is *Pareto-optimal*, *individually-rational*, and *consistent* (see, for instance, Kojima and Ünver, 2014). Hence, we have *Pareto-optimality*, *individual-rationality*, and *consistency* among new applicants. Since the underlying priority structure for the immediate acceptance algorithm used here is a serial dictatorship ordering, it is easy to see that we also have *reallocation-proofness* among new applicants.

Given the sequentiality of rule $\widehat{\phi}$, first using rule $\varphi^{\widehat{\pi}}$ for tenants and then rule $IA^{\widehat{\pi}}$ for new applicants, it follows that $\widehat{\phi}$ satisfies *Pareto-optimality*, *individual-rationality*, *reallocation-proofness*, and *consistency*. However, it is well-known that immediate acceptance rules are *not strategy-proof*. Hence, rule $\widehat{\phi}$ is not *strategy-proof*. □

Consistency. By Proposition 1, a TTC rule based on a cyclic priority structure that is adapted to the ownership structure satisfies *individual rationality*, *Pareto-optimality*, *strategy-proofness*, and *reallocation-proofness*. By Theorem 1 it violates *consistency*. □

Reallocation-Proofness. We consider the case where the set of potential agents is $\mathcal{I} = \{i, j, k\}$ and the set of potential houses is $\mathcal{H} = \{h, h(j)\}$ where h is vacant and $h(j)$ is agent j's house. Let π be a priority structure defined by

$$\pi^h(i) < \pi^h(j) < \pi^h(k)$$

$$\pi^{h(j)}(j) < \pi^{h(j)}(k) < \pi^{h(j)}(i).$$

Let $\widetilde{\pi}$ be a priority structure with $\widetilde{\pi}^{h(j)} = \pi^{h(j)}$ but

$$\widetilde{\pi}^h(j) < \widetilde{\pi}^h(i) < \widetilde{\pi}^h(k).$$

Consider the rule that assigns for each problem (I, H, R),

$$\widetilde{\phi}(I, H, R) = \begin{cases} \varphi^{\widetilde{\pi}}(I, H, R), & \text{if } k \in I, h(j) \in H, \text{ and } h(j)\ P_k\ h_0 \\ \varphi^{\pi}(I, H, R), & \text{otherwise,} \end{cases}$$

that is, if agent k and house $h(j)$ are present and agent k finds house $h(j)$ acceptable, we use the TTC rule based on $\widetilde{\pi}$ and otherwise we use the TTC rule based on π. Note that the two rules φ^{π} and $\varphi^{\widetilde{\pi}}$ are *Pareto-optimal*, *individually-rational*, and *strategy-proof*. This immediately implies that $\widetilde{\phi}$ is *Pareto-optimal*, *individually-rational* and can only be manipulated by agent k. We show that $\widetilde{\phi}$ is *strategy-proof* and *consistent*.

In order to show that also agent k can never improve by changing his preferences, it is sufficient to check the case where $I = \{i, j, k\}$ and $H = \{h, h(j)\}$ since for all other problems with $k \in I$, we have $\widetilde{\phi} = \varphi^{\pi}$ and φ^{π} is *strategy-proof*.

In the following we always have $I = \{i, j, k\}$ and $H = \{h, h(j)\}$ and consider a preference profile $R = (R_i, R_j, R_k)$ and an alternative preference relation R'_k for agent k. We have three cases: either $\widetilde{\phi}_k(R_i, R_j, R'_k) = h_0$ or $\widetilde{\phi}_k(R_i, R_j, R'_k) = h$ or $\widetilde{\phi}_j(R_i, R_j, R'_k) = h(j)$. In the first case ($\widetilde{\phi}_k(R_i, R_j, R'_k) = h_0$), by *individual-rationality*, we have

$$\widetilde{\phi}_k(R)\ R_k\ h_0 = \widetilde{\phi}_k(R_i, R_j, R'_k).$$

The second case ($\widetilde{\phi}_k(R_i, R_j, R'_k) = h$) can only happen if both i and j find h unacceptable under R_i and R_j since agent k is the lowest priority agent under both π^h and $\widetilde{\pi}^h$. Hence, by *Pareto-optimality*,

$$\widetilde{\phi}_k(R)\ R_k\ h = \widetilde{\phi}_k(R_i, R_j, R'_k).$$

In the third case ($\widetilde{\phi}_j(R_i, R_j, R'_k) = h(j)$), by *strategy-proofness* of the two rules φ^{π} and $\varphi^{\widetilde{\pi}}$, we only have to check the case where (i) $h_0\ P_k\ h(j)$ but $h(j)\ P'_k\ h_0$ and the case where (ii) $h(j)\ P_k\ h_0$ but $h_0\ P'_k\ h(j)$. In Case (i),

$$\widetilde{\phi}_k(R)\ R_k\ h_0\ P_k\ h(j) = \widetilde{\phi}_k(R_i, R_j, R'_k).$$

In Case (ii),

$$h_0\ P'_k\ h(j) = \widetilde{\phi}_k(R_i, R_j, R'_k),$$

contradicting *individual-rationality*.

Next we show that rule $\widetilde{\phi}$ is *consistent*. For problems with $|I| \leq 2$ or $|H| = 1$ it is straightforward to see that rule $\widetilde{\phi}$ is *consistent*. Hence, let $I = \{i, j, k\}$ and $H = \{h, h(j)\}$. For preference profiles R with $h(j)\ P_j\ h$ it is straightforward to see that rule $\widetilde{\phi}$ is *consistent* (since agent j will receive $h(j)$ and agent i has a higher priority than agent k in both priority orderings π^h and $\widetilde{\pi}^h$). Thus, consider R with $h\ P_j\ h(j)$. We have two cases depending on whether $h_0\ P_k\ h(j)$ or $h(j)\ P_k\ h_0$.

Case 1. Since $h_0\ P_k\ h(j)$, $\widetilde{\phi} = \varphi^{\pi}$ and

$$\widetilde{\phi}_i(I, H, R) = h, \quad \widetilde{\phi}_j(I, H, R) = h(j), \quad \widetilde{\phi}_k(I, H, R) = h_0.$$

Case 2. Since $h(j)\ P_k\ h_0$, $\widetilde{\phi} = \varphi^{\widetilde{\pi}}$ and

$$\widetilde{\phi}_i(I, H, R) = h_0, \quad \widetilde{\phi}_j(I, H, R) = h, \quad \widetilde{\phi}_k(I, H, R) = h(j).$$

In Case 1, the following reduced problems and their allocations establish *consistency* when one of the agents leaves with his allotment.

$$\begin{aligned}
&\widetilde{\phi}_j(\{j,k\}, \{h(j)\}, (R_j, R_k)) = h(j), \quad \widetilde{\phi}_k(\{j,k\}, \{h(j)\}, (R_j, R_k)) = h_0 \\
&\widetilde{\phi}_i(\{i,k\}, \{h\}, (R_i, R_k)) = h, \quad \widetilde{\phi}_k(\{i,k\}, \{h\}, (R_i, R_k)) = h_0 \\
&\widetilde{\phi}_i(\{i,j\}, \{h, h(j)\}, (R_i, R_j)) = h, \quad \widetilde{\phi}_j(\{i,j\}, \{h, h(j)\}, (R_i, R_j)) = h(j).
\end{aligned}$$

Again, *consistency* easily follows for further reductions from the above two agent problems.

In Case 2, the only reduced problems when one of the agents leaves with his allotment are $(\{j,k\}, \{h, h(j)\}, (R_j, R_k))$ and $(\{i,k\}, \{h(j)\}, (R_i, R_k))$ and their allocations establish *consistency* as follows.

$$\begin{aligned}
&\widetilde{\phi}_j(\{j,k\}, \{h, h(j)\}, (R_j, R_k)) = h, \quad \widetilde{\phi}_k(\{j,k\}, \{h, h(j)\}, (R_j, R_k)) = h(j) \\
&\widetilde{\phi}_i(\{i,k\}, \{h(j)\}, (R_i, R_k)) = h_0, \quad \widetilde{\phi}_k(\{i,k\}, \{h(j)\}, (R_i, R_k)) = h(j).
\end{aligned}$$

Again, *consistency* easily follows for further reductions from the above two agent problems.

Finally, we show that the rule $\widetilde{\phi}$ is not *reallocation-proof*. We consider the problem (I, H, R) with $I = \{i, j, k\}$, $H = \{h, h(j)\}$ and the following preferences:

- $R_i : h\ P_i\ h_0\ P_i \ldots,$
- $R_j : h(j)\ P_j \ldots,$
- $R_k : h\ P_k\ h_0\ P_k \ldots,$
- $\widetilde{R}_j : h\ P_j\ h(j)\ P_j \ldots,$
- $\widetilde{R}_k : h(j)\ P_k\ h_0\ P_k \ldots.$

Let $R = (R_i, R_j, R_k)$ and $\widetilde{R} = (R_i, \widetilde{R}_j, \widetilde{R}_k)$. Then,

$$\begin{aligned} &\widetilde{\phi}_i(I,H,R) = h, \quad \widetilde{\phi}_j(I,H,R) = h(j), \quad \widetilde{\phi}_k(I,H,R) = h_0 \\ &\widetilde{\phi}_i(I,H,\widetilde{R}) = h_0, \quad \widetilde{\phi}_j(I,H,\widetilde{R}) = h, \quad \widetilde{\phi}_k(I,H,\widetilde{R}) = h(j). \end{aligned}$$

We now show that agents j and k by changing their preferences from (R_j, R_k) at R to $(\widetilde{R}_j, \widetilde{R}_k)$ at $\widetilde{R}$ cause a violation of *reallocation-proofness*.

Consider agent j changing his preferences at R from R_j to $\widetilde{R}_i$. For the resulting preference profile $R^1 = (R_i, \widetilde{R}_j, R_k)$ we have $\widetilde{\phi}_j(I,H,R^1) = \widetilde{\phi}_j(I,H,R) = h(j)$. Hence, agent j does not change his allotment by unilaterally moving from R to R^1.

Consider agent k changing his preferences at R from R_k to $\widetilde{R}_k$. For the resulting preference profile $R^2 = (R_i, R_j, \widetilde{R}_k)$ we have $\widetilde{\phi}_k(I,H,R^2) = \widetilde{\phi}_k(I,H,R) = h_0$. Hence, agent k does not change his allotment by unilaterally moving from R to R^2.

Finally, consider both agents j and k changing their preferences at the same time, moving from R to $\widetilde{R}$, and then swapping their allotments. Then, agent j receives the same allotment $\phi_k(I,H,\widetilde{R}) = h(j) = \phi_j(I,H,R)$ while agent k is better off receiving $\phi_j(I,H,\widetilde{R}) = h \, P_k \, h_0 = \phi_k(I,H,R)$; a contradiction to *reallocation-proofness*. □

C Literature Review

C.1 Related Literature: Housing Markets

Shapley and Scarf (1974) introduced housing markets and showed that the (weak) core of a housing market[19] is nonempty. They also defined Gale's Top Trading Cycles (TTC) algorithm (attributed to David Gale). Roth and Postlewaite (1977, Theorem 2') showed that the core of a housing market is unique and it is the outcome of the TTC algorithm.[3] Hence, the core is a rule that selects a matching for each housing market. Roth (1982) showed that the core is *strategy-proof*, and Bird (1984) showed that it is also *group strategy-proof*.

Ma (1994, Theorem 1) showed that the core of a housing market is characterized by *Pareto-optimality*, *individual-rationality (for tenants)*, and *strategy-proofness*; see also Sönmez (1999, Corollary 3) and Svensson (1999, Theorem 2). Sönmez (1996, Theorems 1 and 2) showed that the core is the unique rule satisfying *Pareto-optimality*, *individual-rationality (for tenants)*, and *Maskin-monotonicity*.[20]

Takamiya (2001, Theorem 3.1) showed that for housing markets a rule is *Maskin monotonic* if and only if it is *group strategy-proof*; furthermore, if a rule is *group strategy-proof* and *onto*[21] then it is *Pareto-optimal* (Takamiya, 2001, Lemma 3.5). These results combined

[19] A matching for a housing market is in the *weak core* (or *weakly core stable*) if no subset of agents can strictly benefit by reallocating their occupied houses among themselves.

[20] A rule is *Maskin-monotonic* if each matching that is selected by the rule and that is (weakly) improved in the preferences of all agents (via a so-called Maskin-monotonic transformation) is still selected by the rule.

[21] A rule is *onto* if for each matching there exist preferences of agents at which the matching is selected by the rule.

with Ma's (1994, Theorem 1) characterization imply that a rule is the core if and only if it is *individually-rational (for tenants)*, *Maskin monotonic*, and *onto* (Takamiya, 2001, Corollary 3.8). Takamiya (2001, Theorem 4.12) also showed that a rule is *Maskin monotonic* if and only if it is *strategy-proof* and *non-bossy*, and obtained another characterization of the core by *individual-rationality (for tenants)*, *strategy-proofness*, *ontoness*, and *non-bossiness* (Takamiya, 2001, Corollary 4.16).

Ehlers (2014) studied housing markets (when agents have weak preferences over houses). He showed that when agents have strict preferences over houses a rule satisfies *weak Pareto-optimality*,[22] *individual-rationality*, *strategy-proofness*, and *consistency* if and only if it is the TTC rule (Ehlers, 2014, Lemma 1). On the weak preference domain, the TTC rule with fixed tie-breaking is characterized by *weak Pareto-optimality*, *individual-rationality*, *strategy-proofness*, *non-bossiness*, and *consistency* (Ehlers, 2014, Theorem 1).

C.2 Related Literature: House Allocation Problems

For house allocation problems, the serial dictatorship rule works as follows: given an ordering of agents, the first agent in the ordering is assigned his most preferred house, the second agent in the ordering is assigned his most preferred house among the remaining houses, and so on. Svensson (1994, Theorem 1) showed that the serial dictatorship rule is *Pareto-optimal* and *strategy-proof*, moreover the set of all *Pareto-optimal* matchings are obtained by applying the serial dictatorship rule at every ordering of agents (Svensson, 1994, Theorem 2). Svensson (1999, Theorem 1) showed that a rule satisfies *strategy-proofness*, *non-bossiness*, and *neutrality*[23] if and only if it is the serial dictatorship rule.

Ergin (2000, Proposition 1) showed that the Pareto correspondence that selects all *Pareto-optimal* matchings for a house allocation problem is *consistent*,[7] and a rule for house allocation problems satisfies *Pareto-optimality*, *neutrality*, and *consistency* if and only if it is a serial dictatorship rule (Ergin, 2000, Theorem 1 and Corollary 1).

Pápai (2000) introduced hierarchical exchange rules for house allocation problems. Hierarchical exchange rules assign houses to agents in a similar way the TTC algorithm does (for housing markets) by specifying ownership rights for the houses in an iterative hierarchial manner. Each house is an initial endowment of an agent and an agent may be endowed with multiple houses. At the first step of the TTC algorithm each agent points to the agent who is endowed with his favorite object. Agents at the top trading cycles are assigned to their favorite houses and removed from the problem with their assignments. Houses that were not assigned to anyone at the first step of the algorithm and were endowments of removed agents are inherited as new endowments by the agents who are still in the problem. Hence each remaining house is an endowment of some remaining agent at the second step of the TTC algorithm. The TTC algorithm can continue and agents in the top trading cycles are

[22] A rule is *weakly Pareto-optimal* if the matching chosen by the rule is such that there is no other matching at which all agents are strictly better off.

[23] A rule for house allocation problems is *neutral* if the matching selected by the rule is independent of the names of the houses.

assigned their favorite houses among the remaining ones and removed from the problem with their assigned houses at every step of the algorithm. Houses that are still in the problem and were endowments of removed agents are inherited by agents who are not removed from the problem at every step of the algorithm. Hence a hierarchical exchange rule is determined by the initial endowments and the hierarchical endowment inheritance at later steps. While the initial endowments are given a priori, the hierarchical endowment inheritance may be endogenous; in particular, the inheritance of endowments may depend on the allotments in earlier steps (for details on the exact "rules" of endowment inheritance we refer to Pápai, 2000, Section 5.1). Pápai (2000) showed that a rule for house allocation problems satisfies *Pareto-optimality*, *group strategy-proofness*, and *reallocation-proofness* if and only if it is a hierarchical exchange rule. A subclass of the hierarchical exchange rules are the so-called endowment inheritance rules (called "fixed endowment hierarchical exchange rules" by Pápai, 2000) that are defined by an endowment inheritance table that shows the initial endowments of agents and the order of inheritance of each house. That is, an endowment inheritance table is a permutation of the agents for each house. Notice that a serial dictatorship rule is an endowment inheritance rule defined by the endowment inheritance table that uses the same permutations of agents for each house (then, at every step of the TTC algorithm, one agent will be endowed with all houses).

Ehlers and Klaus (2007, Theorem 1 and Corollary 1) showed that if a rule for house allocation problems with at least four agents satisfies *Pareto-optimality*, *strategy-proofness*, and *consistency* then it is a so-called efficient generalized priority rule, i.e., it adapts to a priority structure that satisfies Ergin's acyclicity condition (Ergin, 2002), except -maybe- for up to three agents in each house's priority ordering. Ehlers and Klaus (2006) studied house allocation problems for endogenously given priority structures. A rule is an efficient priority rule if it adapts to an Ergin acyclic priority structure and the assignment of houses to agents are determined by the agents-proposing deferred acceptance rule. They showed that rules for house allocation problems satisfy *Pareto-optimality*, *group strategy-proofness*, and *reallocation-consistency*[10] if and only if they are efficient priority rules (Ehlers and Klaus, 2006, Proposition 2 and Theorem 1).

Ehlers et al. (2002) studied house allocation problems when the population changes. They characterized rules that satisfy *Pareto-optimality*, *strategy-proofness*, and *population-monotonicity*[24] (Ehlers et al., 2002, Theorem 1). The characterized rules are restricted endowment inheritance rules that assign houses to agents by an iterative procedure such that at each step no more than two agents trade houses from their hierarchically specified endowments. Ehlers and Klaus (2004) (see also Ehlers and Klaus, 2011 and Kesten, 2009) studied house allocation problems when houses (resources) change. They characterized rules that satisfy *Pareto-optimality*, *independence of irrelevant objects*,[25] and *resource monotonic-*

[24]A rule for house allocation problems is *population-monotonic* when some agents are added to the set of incumbent agents then all incumbent agents are either (weakly) worse off or (weakly) better off.

[25]A rule satisfies *independence of irrelevant objects* if the matching chosen by the rule depends only on preferences over the set of available houses, i.e., when two preferences over a given set of houses coincide then the rule chooses the same matching at these preferences. The condition is vacuously satisfied in our

ity[26] when the null house is always the worst house for each agent (Ehlers and Klaus, 2004, Theorem 1). The characterized rules are the mixed dictator-pairwise exchange rules that assign houses to agents sequentially such that at each step there is either a dictator or two agents trade objects from their hierarchically specified endowments. Efficient priority rules characterized by Ehlers and Klaus (2006), restricted endowment inheritance rules characterized by Ehlers et al. (2002), and mixed dictator-pairwise exchange rules characterized by Ehlers and Klaus (2004) are the same rules. That is, they are endowment inheritance rules (Pápai, 2000) defined by the endowment inheritance tables such that at each step of the TTC algorithm all remaining houses are endowed either by an agent (the dictator) or by two agents.

C.3 Related Literature: Acyclic Priority Structures

Ergin (2002) and Kesten (2006) studied a more general *house allocation with quotas model*: There is a finite set of houses that each come with a quota that describes the number of available copies to be allocated to a finite set of agents. Each agent has strict preferences over houses and each house has a fixed priority ordering over agents. We refer to such a problem as house allocation with quotas problem but a well-known application is that of a *school choice problem* where each house with a quota is a school with seats, each agent is a student who wants to be enrolled in a school, and the priority ordering of a school is determined by law or other criteria by the schools or school districts (two surveys on school choice can be found in Abdulkadiroğlu, 2013; Pathak, 2011). The only difference between house allocation problems and house allocation with quotas problems is that the quota of every house is exactly one in the former and it is at least one in the latter.

Ergin (2002) defined an acyclicity condition on priority structures for house allocation with quotas problems that we refer to as *Ergin acyclicity*. Let (I, H, R) be a problem, $q_H = (q_h)_{h\in H}$ denote the quotas of houses in H, and π_I^H denote a priority structure for houses in H over agents in I. For each house $h \in H$ and each agent $i \in I$, let $U_h(i) = \{j \in I \mid \pi^h(j) < \pi^h(i)\}$ denote the set of agents who have higher priority than agent i for house h. A priority structure (π_I^H) has an *Ergin cycle* if the following conditions are satisfied:

- (*Ergin's Cycle Condition*). There are agents $i, j, k \in I$ and houses $h, h' \in H$ such that $\pi^h(i) < \pi^h(j) < \pi^h(k)$ and $\pi^{h'}(k) < \pi^{h'}(i)$.
- (*Ergin's Scarcity Condition*). There are (possibly empty) disjoint sets for houses h and h', $I_h, I_{h'} \subseteq I \setminus \{i, j, k\}$ such that $I_h \subseteq U_h(j)$, $I_{h'} \subseteq U_{h'}(i)$, $|I_h| = q_h - 1$, and $|I_{h'}| = q_{h'} - 1$.

A priority structure is *Ergin acyclic* if it has no Ergin cycles. Note that Ergin acyclicity imposes conditions both on the priority structure and the quotas of houses. If the quota

formulation of the problem.

[26] A rule is *resource-monotonic* if the following holds: if more resources become available, i.e., a set of new houses is added to existing houses, then all agents either (weakly) gain or (weakly) loose.

of each house is one, then Ergin's scarcity condition is automatically satisfied and Ergin acyclicity only restricts the priority structure through Ergin's cycle condition.

Ergin (2002) showed that the agents-proposing deferred acceptance rule (Gale and Shapley, 1962) based on a priority structure π, denoted by DA^π, is a so-called "best rule," i.e., it *Pareto-dominates* any other rule that adapts to the same priority structure (Ergin, 2002, Proposition 1). He also showed that the following are equivalent: DA^π is *Pareto-optimal*, DA^π is *group strategy-proof*, DA^π is *consistent*, and π is Ergin acyclic (Ergin, 2002, Theorem 1).

Kesten (2006) defined an acyclicity condition on priority structures for house allocation with quotas problems that we refer to as *Kesten acyclicity*. Let (I, H, R) be a problem, $q_H = (q_h)_{h \in H}$ denote the quotas of houses in H, and π_I^H denote a priority structure for houses in H over agents in I. A priority structure (π_I^H) has a *Kesten cycle* if the following conditions are satisfied:

- (*Kesten's Cycle Condition*). There are agents $i, j, k \in I$ and houses $h, h' \in H$ such that $\pi^h(i) < \pi^h(j) < \pi^h(k)$, $\pi^{h'}(k) < \pi^{h'}(i)$, and $\pi^{h'}(k) < \pi^{h'}(j)$.
- (*Kesten's Scarcity Condition*). There is a (possibly empty) set for house h, $I_h \subseteq I \setminus \{i, j, k\}$ such that $I_h \subseteq U_h(i) \cup \big(U_h(j) \setminus U_{h'}(k)\big)$ and $|I_h| = q_h - 1$.

A priority structure is *Kesten acyclic* if it has no Kesten cycles.

Similar to Ergin's acyclicity condition, Kesten acyclicity imposes conditions both on the priority structure and the quotas of houses. If the quota of each house is one, then Kesten's scarcity condition is automatically satisfied, and Kesten acyclicity only restricts the priority structure through Kesten's cycle condition. Note that for house allocation with quotas problems, Kesten acyclicity is stronger than Ergin acyclicity (Kesten, 2006, Lemma 1), i.e., if a priority structure has an Ergin cycle then it also has a Kesten cycle.

Kesten (2006) showed that for the TTC rule based on a priority structure π,[27] denoted by φ^π, the following are equivalent: $\varphi^\pi = DA^\pi$, φ^π is *resource-monotonic*,[26] φ^π is *population-monotonic*,[24] and π is Kesten acyclic (Kesten, 2006, Theorem 1).

Kesten (2006) defined another acyclicity condition on priority structures for house allocation with quotas problems that we refer to as *strong Kesten acyclicity*. Let (I, H, R) be a problem, $q_H = (q_h)_{h \in H}$ denote the quotas of houses in H, and π_I^H denote a priority structure for houses in H over agents in I. A priority structure (π_I^H) has a *weak Kesten cycle* if Kesten's cycle condition applies and the following condition is satisfied:

- (*Kesten's Weak Scarcity Condition*). There is a (possibly empty) set for house h, $I_h \subseteq I \setminus \{i, j, k\}$ such that $I_h \subseteq U_h(k)$ and $|I_h| = q_h - 1$.

[27] In the TTC algorithm, each agent is assigned the house he points to in a trading cycle and the quota of each house in a trading cycle is reduced by one. Furthermore, if the quota of a house becomes zero at the end of some step of the TTC algorithm, then the house is removed.

A priority structure is *strongly Kesten acyclic* if it has no weak Kesten cycles.

If a priority structure satisfies strong Kesten acyclicity, then it also satisfies Kesten acyclicity. If the quota of each house is one, then Kesten's weak scarcity condition is automatically satisfied, and both of Kesten's acyclicity conditions coincide.

Kesten (2006) showed that the TTC rule based on priority structure π, φ^{π}, is *consistent* if and only if π is strongly Kesten acyclic (Kesten, 2006, Theorem 2).

For completeness, we show that for house allocation problems with only new applicants, Ergin and Kesten acyclicity coincide.

Lemma 6 (**Ergin = Kesten Acyclicity for House Allocation Problems**)**.** *For house allocation problems with only new applicants, a priority structure is Ergin acyclic if and only if it is (strongly) Kesten acyclic.*

***Proof*.** For house allocation problems, Ergin's as well as Kesten's scarcity conditions of acyclicity are satisfied automatically. Furthermore, it is easy to see that when Kesten's cycle condition is satisfied, then Ergin's cycle condition is satisfied as well, i.e., a Kesten cycle is also an Ergin cycle.

We now show that any Ergin cycle implies the existence of a Kesten cycle. Assume that Ergin's cycle condition is satisfied, i.e., there are agents $i, j, k \in I$ and houses $h, h' \in H$ such that $\pi^{h}(i) < \pi^{h}(j) < \pi^{h}(k)$ and $\pi^{h'}(k) < \pi^{h'}(i)$. If $\pi^{h'}(k) < \pi^{h'}(j)$, then Kesten's cycle condition is satisfied and we are done. Hence, assume that $\pi^{h'}(k) > \pi^{h'}(j)$. Then, $\pi^{h'}(j) < \pi^{h'}(k) < \pi^{h'}(i)$, $\pi^{h}(i) < \pi^{h}(j)$, and $\pi^{h}(i) < \pi^{h}(k)$; hence, Kesten's cycle condition is satisfied. $\square$

References

Abdulkadiroğlu, A. (2013): "School Choice." In N. Vulkan, A. E. Roth, and Z. Neeman, editors, *The Handbook of Market Design*, chapter 5, pages 138–169. Oxford University Press.

Abdulkadiroğlu, A. and Sönmez, T. (1999): "House Allocation with Existing Tenants." *Journal of Economic Theory*, 88: 233–260.

Balinski, M. and Sönmez, T. (1999): "A Tale of Two Mechanisms: Student Placement." *Journal of Economic Theory*, 84: 73–94.

Bird, C. G. (1984): "Group Incentive Compatibility in a Market with Indivisible Goods." *Economics Letters*, 14: 309–313.

Ehlers, L. (2014): "Top Trading with Fixed Tie-Breaking in Markets with Indivisible Goods." *Journal of Economic Theory*, 151: 64–87.

Ehlers, L. and Klaus, B. (2004): "Resource-Monotonicity for House Allocation Problems." *International Journal of Game Theory*, 32: 545–560.

Ehlers, L. and Klaus, B. (2006): “Efficient Priority Rules.” *Games and Economic Behavior*, 55: 372–384.

Ehlers, L. and Klaus, B. (2007): “Consistent House Allocation.” *Economic Theory*, 30: 561–574.

Ehlers, L. and Klaus, B. (2011): “Corrigendum to “Resource-Monotonicity for House Allocation Problems”.” *International Journal of Game Theory*, 40: 281–287.

Ehlers, L., Klaus, B., and Pápai, S. (2002): “Strategy-Proofness and Population-Monotonicity for House Allocation Problems.” *Journal of Mathematical Economics*, 38: 329–339.

Ergin, H. I. (2000): “Consistency in House Allocation Problems.” *Journal of Mathematical Economics*, 34: 77–97.

Ergin, H. I. (2002): “Efficient Resource Allocation on the Basis of Priorities.” *Econometrica*, 70: 2489–2497.

Gale, D. and Shapley, L. S. (1962): “College Admissions and the Stability of Marriage.” *The American Mathematical Monthly*, 69: 9–15.

Hylland, A. and Zeckhauser, R. (1979): “The Efficient Allocation of Individuals to Positions.” *Journal of Political Economy*, 87: 293–314.

Kesten, O. (2006): “On Two Competing Mechanisms for Priority-Based Allocation Problems.” *Journal of Economic Theory*, 127: 155–171.

Kesten, O. (2009): “Coalitional Strategy-Proofness and Resource Monotonicity for House Allocation Problems.” *International Journal of Game Theory*, 38: 17–21.

Kojima, F. and Ünver, M. U. (2014): “The “Boston” School Choice Mechanism: An Axiomatic Approach.” *Economic Theory*, 55: 515–544.

Ma, J. (1994): “Strategy-Proofness and the Strict Core in a Market with Indivisibilities.” *International Journal of Game Theory*, 23: 75–83.

Pápai, S. (2000): “Strategyproof Assignment by Hierarchical Exchange.” *Econometrica*, 68: 1403–1433.

Pathak, P. A. (2011): “The Mechanism Design Approach to Student Assignment.” *Annual Reviews of Economics*, 3: 513–536.

Pycia, M. and Ünver, M. U. (2017): “Incentive Compatible Allocation and Exchange of Discrete Resources.” *Theoretical Economics*, 12: 287–329.

Roth, A. E. (1982): “Incentive Compatibility in a Market with Indivisible Goods.” *Economics Letters*, 9: 127–132.

Roth, A. E. and Postlewaite, A. (1977): "Weak versus Strong Domination in a Market with Indivisible Goods." *Journal of Mathematical Economics*, 4: 131–137.

Satterthwaite, M. and Sonnenschein, H. (1981): "Strategy-Proof Allocation Mechanisms at Differentiable Points." *The Review of Economic Studies*, 48: 587–597.

Shapley, L. S. and Scarf, H. (1974): "On Cores and Indivisibility." *Journal of Mathematical Economics*, 1: 23–37.

Sönmez, T. (1996): "Implementation in Generalized Matching Problems." *Journal of Mathematical Economics*, 26: 429–439.

Sönmez, T. (1999): "Strategy-Proofness and Essentially Single-Valued Cores." *Econometrica*, 67: 677–689.

Sönmez, T. and Ünver, M. U. (2010): "House Allocation with Existing Tenants: A Characterization." *Games and Economic Behavior*, 69: 425–445.

Svensson, L.-G. (1994): "Queue Allocation of Indivisible Goods." *Social Choice and Welfare*, 11: 323–330.

Svensson, L.-G. (1999): "Strategy-Proof Allocation of Indivisible Goods." *Social Choice and Welfare*, 16: 557–567.

Takamiya, K. (2001): "Coalition Strategy-Proofness and Monotonicity in Shapley-Scarf Housing Markets." *Mathematical Social Sciences*, 41: 201–213.

Thomson, W. (2016): "Non-Bossiness." *Social Choice and Welfare*, 47: 665–696.